Interdisciplinary
Mathematics
Institute

2012:09

Hilbert spaces of vector-valued functions generated by quadratic forms

K.S. Kazarian and V.N. Temlyakov

IMI
Preprint Series

College of Arts and Sciences
University of South Carolina

Hilbert spaces of vector-valued functions generated by quadratic forms

K.S. Kazarian[a,*], V.N. Temlyakov[b]

[a]*Dept. of Mathematics, Universidad Autónoma de Madrid,28049, Madrid, Spain e-mail: kazaros.kazarian@uam.es*

[b]*Dept. of Mathematics, 1523 Greene Street, University of South Carolina, Columbia, SC 29208, USA e-mail: temlyakovv@gmail.com*

Abstract

We study Hilbert spaces $\mathfrak{L}^2(E,G)$, where $E \subset \mathbb{R}^d$ is a measurable set, $|E| > 0$ and for almost every $t \in E$ the matrix $G(t)$ (see (3)) is a Hermitian positive-definite matrix. We find necessary and sufficient conditions for which the projection operators $T_k(f)(\cdot) = f_k(\cdot)\mathbf{e}_k, 1 \le k \le n$ are bounded. The obtained results allow us to translate various questions in the spaces $\mathfrak{L}^2(E,G)$ to weighted norm inequalities with weights which are the diagonal elements of the matrix $G(t)$. In Section 3 we study the properties of the system $\{\varphi_m(t)\mathbf{e}_j, 1 \le j \le n; m \in \mathbb{N}\}$ in the space $\mathfrak{L}^2(E,G)$, where $\Phi = \{\varphi_m\}_{m=1}^{\infty}$ is a complete orthonormal system defined on a measurable set $E \subset \mathbb{R}$. We concentrate our study on two classical systems: the Haar and the trigonometric systems. Simultaneous approximations of n elements $F_1, \dots, F_n$ of some Banach spaces $X_1, \dots, X_n$ with respect to a system Ψ which is a basis in any of those spaces are studied.

Keywords: Hilbert space, vector-valued function, Hermitian form, bounded projection, basis, unconditional basis, weighted-norm space, greedy basis, simultaneous approximation

2010 MSC: 46C05, 46E30, 42C10, 41A30, 41A46

---

1. Introduction

The theory of weighted spaces is well developed in the case of scalar functions. A standard weighted space in this regard is the $L^p(w)$ defined

*Corresponding author: K.S.Kazarian Tel.: 34-91 4974489; Fax: 34-91 4974889

as follows. Let w be a nonnegative integrable on $[0,1]$ function. Define for $1 \le p < \infty$

$$\|f\|_{p,w}^p := \int_0^1 |f(t)|^p w(t) dt.$$

The main interest of this paper is in consideration of weighted spaces of vector-valued functions, say, functions of the form $f = (f_1, \dots, f_d)$, where f_j are scalar functions. A strait forward way to generalize a scalar setting to the vector-valued setting would be as follows. Introduce d weights $W := (w_1, \dots, w_d)$ and define

$$\|f\|_{p,W} := \left(\sum_{j=1}^d \|f_j\|_{p,w_j}^2 \right)^{1/2}. \tag{1}$$

This definition corresponds to a diagonal matrix with diagonal elements $(w_1, \dots, w_d)$. We opt for a definition based on a given Hermitian positive-definite form rather than on a diagonal matrix. We consider here only the case $p = 2$ and define the norm corresponding to a given Hermitian positive-definite form with a matrix G as follows

$$\|f\|_{\mathfrak{L}^2(E,G)} := \left(\int_E f^*(t) G(t) f(t) dt \right)^{1/2}. \tag{2}$$

Clearly, definition (2) is more general than definition (1). It turns out that this generality brings about some unexpected phenomena. For instance, it is obvious that in the case of (1) the projection operator that maps f into $(0, \dots, 0, f_j, 0 \dots, 0)$ is a bounded operator. We prove in Section 2 that it is not always the case in the space $\mathfrak{L}^2(E,G)$.

We study Hilbert spaces of vector-valued functions $\mathfrak{L}^2(E,G)$, where $E \subset \mathbb{R}^d$ is a measurable set, $|E| > 0$ and for almost every $t \in E$ the matrix

$$G(t) = \Big(g_{jk}(t) \Big)_{1 \le j,k \le n} \tag{3}$$

is a Hermitian positive-definite (HPD) matrix. Similar spaces appeared in the literature earlier (e.g. see [1]). Our approach is based on a Hermitian positive-definite matrix G. In Section 2 we find necessary and sufficient conditions (see Theorem 2.1) for the above described projection operators to be bounded. Those conditions are fundamental for the proof of results obtained in Sections 3 and 4. Proposition 2.2 which is based on Theorem 2.1 allows us to translate various questions in the spaces $\mathfrak{L}^2(E,G)$ to weighted norm inequalities with weights which are the diagonal elements of the matrix $G(t)$. In Section 3 we study the properties of the system $\{\varphi_m(t)\mathbf{e}_j,\ 1\le j\le n; m\in\mathbb{N}\}$ in the space $\mathfrak{L}^2(E,G)$, where $\Phi=\{\varphi_m\}_{m=1}^{\infty}$ is a complete orthonormal system defined on a measurable set $E\subset\mathbb{R}$. We concentrate our study on two classical systems: the Haar and the trigonometric systems. M. Nielsen [9] has studied a similar question for the trigonometric system. Conditions obtained in the cited article are different because the questions that have been considered are different. In [9] the author in fact has studied the question of being a summation basis of the considered system.

The obtained results are applied in the next section to study simultaneous approximation of n elements $F_1,\dots,F_n$ of some Banach spaces $X_1,\dots,X_n$ with respect to a system Ψ which is a basis in each of those spaces. We discuss there the properties *greedy* and *democratic* of a basis which are important in nonlinear sparse approximation.

2. General $\mathfrak{L}^2(E,G)$ spaces

2.1. Hermitian forms

Let V be a vector space over a field $\mathbb{K}(\mathbb{K}=\mathbb{C}\,\mathrm{or}\,\mathbb{R})$ and let $\dim V=n$. A transformation $\Phi: V\times V\to\mathbb{K}$ is called a Hermitian form if

$$\begin{aligned}\Phi(\alpha\mathbf{u}+\beta\mathbf{v},\mathbf{w}) &= \alpha\Phi(\mathbf{u},\mathbf{w})+\beta\Phi(\mathbf{v},\mathbf{w});\\ \Phi(\mathbf{w},\alpha\mathbf{u}+\beta\mathbf{v}) &= \bar{\alpha}\Phi(\mathbf{w},\mathbf{u})+\bar{\beta}\Phi(\mathbf{w},\mathbf{v});\\ \Phi(\mathbf{u},\mathbf{v}) &= \overline{\Phi(\mathbf{v},\mathbf{u})}\end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$ and $\alpha,\beta\in\mathbb{K}$. If $\{\mathbf{e}_j\}_{j=1}^n$ is a basis of V then for

$$\mathbf{x}=\sum_{j=1}^n x_j\mathbf{e}_j;\qquad \mathbf{y}=\sum_{j=1}^n y_j\mathbf{e}_j$$

we will have that

$$\Phi(\mathbf{x},\mathbf{y})=\sum_{j=1}^n\sum_{k=1}^n x_j\bar{y}_k\Phi(\mathbf{e}_j,\mathbf{e}_k)=\sum_{j,k}\Phi_{jk}x_j\bar{y}_k.$$

The Hermitian matrix

$$\Phi=(\Phi_{jk})_{1\le j,k\le n}\,,\qquad \Phi_{jk}=\bar{\Phi}_{kj}$$

is called the matrix of the given Hermitian form with respect to the basis $\{\mathbf{e}_j\}_{j=1}^n$ of the space V. If $\{\mathbf{e}'_j\}_{j=1}^n$ is another basis of the vector space V and

$$\mathbf{e}'_j = \sum_{k=1}^{n} a_{kj}\mathbf{e}_k, \qquad 1 \le j \le n.$$

Then the matrix Φ' of the Hermitian form $\Phi(\cdot,\cdot)$ with respect to the basis $\{\mathbf{e}'_j\}_{j=1}^n$ is defined by the following formula

$$\Phi' = A^* \cdot \Phi \cdot A,$$

where $A = (a_{kj})_{1\le k,j\le n}$ and A^* is the conjugate transpose of A, obtained from A by taking transpose and then taking the complex conjugate.

The quadratic form $\Phi(\mathbf{x},\mathbf{x})$ is called Hermitian positive-definite if

$$\Phi(\mathbf{x},\mathbf{x}) = \sum_{j,k} \Phi_{jk} x_j \bar{x}_k > 0 \qquad \text{for all}\, x_j \in \mathbb{K}, 1 \le j \le n$$

such that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is nonzero. In this case the matrix Φ is called Hermitian positive-definite (HPD) matrix. Recall some properties of HPD matrices:

Every HPD matrix is invertible and its inverse is also HPD matrix.

If Φ is a HPD matrix then the diagonal entries Φ_{jj} are real and positive. As a consequence the trace, $\mathrm{tr}(\Phi) > 0$. Furthermore

$$|\Phi_{jk}| \le \sqrt{\Phi_{jj}\Phi_{kk}}, \quad 1 \le j,k \le n. \tag{4}$$

2.2. General $\mathfrak{L}^2(E,G)$ spaces

Let $E \subset \mathbb{R}^d$ be a measurable set, $|E| > 0$. A $V-$valued function $\eta : E \to V$ will be called measurable if it is defined by the equation $\eta(t) = \sum_{k=1}^n \alpha_k(t)\mathbf{e}_k$, where $\alpha_k : E \to \mathbb{K}, 1 \le k \le n$ are Lebesgue measurable functions. Let $\mathfrak{B} : E \to \mathbb{K}^{n\times n}$ be a matrix-valued function defined on E such that for any $t \in E$ the matrix

$$\mathfrak{B}(t) = \Big(\beta_{kj}(t)\Big)_{1\le k,j\le n}$$

is a nonsingular matrix and the functions β_{kj} are measurable on E. We define $V-$valued measurable functions

$$\mathbf{e}_j(t) = \sum_{k=1}^{n} \beta_{kj}(t)\mathbf{e}_k, \qquad 1 \le j \le n \tag{5}$$

and observe that for any $t \in E$ $\{\mathbf{e}_j(t)\}_{j=1}^n$ is a basis of the vector space V. Such a basis we will call measurable basis of V.

For any $t \in E$ let

$$\mathfrak{B}^{-1}(t) = \left(\beta_{kj}^{(-1)}(t)\right)_{1 \leq k,j \leq n}$$

be the inverse of the matrix $\mathfrak{B}(t)$. The functions $\beta_{kj}^{(-1)}$ are measurable on E. It follows from the Kramer's rule.

Let for any $t \in E$ $G_t : V \times V \to \mathbb{K}$ be a Hermitian form such that the functions

$$g_{jk}(t) = G_t(\mathbf{e}_j(t), \mathbf{e}_k(t)) \qquad 1 \leq j, k \leq n$$

are measurable and the matrix

$$G(t) = \left(g_{jk}(t)\right)_{1 \leq j,k \leq n} \tag{6}$$

is a HPD matrix a. e. on E. Observe that by properties formulated above

$$g_{kk}(t) \geq 0 \qquad \text{a. e. on } E, 1 \leq k \leq n. \tag{7}$$

The HPD matrices are invertible, hence, for a. e. $t \in E$ the matrix $G(t)$ is invertible. Let

$$G^{-1}(t) = \left(g_{jk}^{(-1)}(t)\right)_{1 \leq j,k \leq n}$$

be the inverse matrix for any $t \in E$ for which $G(t)$ is HPD. As it was shown above the functions $g_{jk}^{(-1)}$ are measurable on E.

If $w \geq 0$ is a measurable function on E then we say that $\phi \in L^2(E, w)$ if $\phi : E \to \mathbb{K}$ is measurable on E and the norm is defined by

$$\|\phi\|_{L^2(E,w)} := \left(\int_E |\phi(t)|^2 w(t) dt\right)^{\frac{1}{2}} < +\infty.$$

Let $\mathfrak{L}$ be the vector space of all V-valued functions $f : E \to V$ such that

$$f(t) = \sum_{k=1}^{n} f_k(t) \mathbf{e}_k(t), \tag{8}$$

where the functions $f_j (1 \leq j \leq n)$ are measurable and $f_j \in L^2(E, g_{jj}), 1 \leq j \leq n$. We will write $f = h$ for $f, h \in \mathfrak{L}$ if and only if $f_j = h_j$ a. e. on E for all $1 \leq j \leq n$.

By (4) we have that

$$|g_{jk}(t)| \le \sqrt{g_{jj}(t) g_{kk}(t)} \qquad \text{a. e. on } E, 1 \le j, k \le n. \tag{9}$$

Furthermore if we write $f(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$ it will be understood that the equation (8) holds. For $f, h \in \mathfrak{L}$ let

$$\langle f|h \rangle_G = \int_E h^*(t) G(t) f(t) dt = \int_E \sum_{j,k} \bar{h}_j(t) g_{jk}(t) f_k(t) dt. \tag{10}$$

By (9) it follows that $\langle f|h \rangle_G$ exists for any $f, h \in \mathfrak{L}$. We define a norm on $\mathfrak{L}$ based upon the inner product

$$\|f\|_{\mathfrak{L}^2(E,G)} := \sqrt{\langle f|f \rangle_G}.$$

Afterwards we consider the closure of $\mathfrak{L}$ in the norm defined above and denote by $\mathfrak{L}^2(E, G)$ the obtained Hilbert space. As usual, we identify $\mathfrak{L}^2(E, G)$ with the quotient $\mathfrak{L}^2(E, G)/L_0$, where

$$L_0 := \{f : E \to V : f(t) = 0 \quad \text{a. e. on } E\}.$$

Observe that the values of a given Hermitian form don't depend on the choice of the basis. Hence the inner product and consequently the norm in $\mathfrak{L}^2(E, G)$ doesn't depend on the choice of the measurable basis of V.

Let for $k \in [1, n]$

$$\eta_k(t) = \langle g_{1k}^{(-1)}(t), g_{2k}^{(-1)}(t), \ldots, g_{nk}^{(-1)}(t) \rangle = \sum_{j=1}^{n} g_{jk}^{(-1)}(t) \mathbf{e}_j(t). \tag{11}$$

The system $\{\eta_k(t)\}_{k=1}^n$ is biorthogonal to $\{\mathbf{e}_k(t)\}_{k=1}^n$. Using the property $g_{jk} = \bar{g}_{kj}$ it is easy to check that $\langle \mathbf{e}_k|\eta_s \rangle_G = \delta_{ks}$. The following statement holds.

Proposition 2.1. *Let $\varphi, \psi : E \to \mathbb{K}$ be measurable functions. Inclusion*

$$\varphi(t)\mathbf{e}_k(t) \in \mathfrak{L}^2(E, G)$$

holds for some k $(1 \le k \le n)$ if and only if $\varphi \in L^2(E, g_{kk})$. Then

$$\|\varphi(\cdot)\mathbf{e}_k(\cdot)\|_{\mathfrak{L}^2(E,G)} = \|\varphi\|_{L^2(E,g_{kk})}.$$

Inclusion

$$\psi(t)\eta_k(t) \in \mathfrak{L}^2(E,G)$$

holds for some k $(1 \le k \le n)$ *if and only if* $\psi \in L^2(E, g_{kk}^{(-1)})$. *Then*

$$\|\psi\eta_k\|_{\mathfrak{L}^2(E,G)} = \|\psi\|_{L^2(E,g_{kk}^{(-1)})}.$$

Proof. We only give the proof of the second assertion. The first one follows from the definition of $\langle\cdot|\cdot\rangle_G$. We have

$$\begin{aligned}\eta_k^*(t)G(t)\eta_k(t) &= \sum_{l=1}^{n}\sum_{j=1}^{n}\overline{g_{lk}^{(-1)}}(t)g_{lj}(t)g_{jk}^{(-1)}(t)\\ &= \sum_{l=1}^{n}\overline{g_{lk}^{(-1)}}(t)\delta_{lk} = g_{kk}^{(-1)}(t),\end{aligned}$$

where δ_{lk} is the Kronecker delta, equal to zero whenever $l \neq k$ and equal to 1 if $l = k$. Hence,

$$\langle\psi\eta_k|\psi\eta_k\rangle_G = \int_E |\psi(t)|^2 g_{kk}^{(-1)}(t)dt.$$

□

Lemma 2.1. *For any* $k, 1 \le k \le n$

$$1 \le g_{kk}(t)g_{kk}^{(-1)}(t) \quad a.\ e.\ on\, E$$

Proof. For any $\varphi \in L^2(E, g_{kk})$, $\psi \in L^2(E, g_{kk}^{(-1)})$ we have that

$$\langle\varphi(\cdot)\mathbf{e}_k(\cdot)|\psi(\cdot)\eta_k(\cdot)\rangle_G = \int_E \varphi(t)\bar{\psi}(t)dt$$

and by the Cauchy-Schwartz inequality

$$\begin{aligned}\left|\int_E \varphi(t)\bar{\psi}(t)dt\right| &\le \|\varphi\mathbf{e}_k\|_{\mathfrak{L}^2(E,G)}\|\psi\eta_k\|_{\mathfrak{L}^2(E,G)}\\ &= \|\varphi\|_{L^2(E,g_{kk})}\|\psi\|_{L^2(E,g_{kk}^{(-1)})}.\end{aligned}$$

The above inequality with $\varphi = \psi(g_{kk})^{-1}$ yields

$$\left(\int_E |\psi(t)|^2 \frac{1}{g_{kk}(t)}dt\right)^{\frac{1}{2}} \le \|\psi\|_{L^2(E,g_{kk}^{(-1)})}.$$

Recall that $g_{kk}(t) > 0$, $g_{kk}^{(-1)}(t) > 0$ a. e. on E. We now apply a well known result on a multiplication operator $T_M : L^2(E) \to L^2(E)$ defined by the equation $T_M(g) = Mg$ is bounded if and only if $M \in L^\infty(E)$ and $\|T_M\| = \|M\|_{L^\infty(E)}$. Hence,

$$\frac{1}{g_{kk}(t)g_{kk}^{(-1)}(t)} \leq 1 \qquad \text{a. e. on}\, E.$$

$\square$

Lemma 2.2. *If for some k $(1 \leq k \leq n)$ there exists $\varphi_0 \in L^2(E, 1/g_{kk}^{(-1)})$ such that $\varphi_0 \notin L^2(E, g_{kk})$ then the transformation*

$$T_k(f)(t) = f_k(t)\mathbf{e}_k(t)$$

is not a bounded projection on $\mathfrak{L}^2(E, G)$.

Proof. Let $h_0(t) = \frac{\varphi_0(t)}{g_{kk}^{(-1)}(t)}$. It is easy to check that $h_0 \in L^2(E, g_{kk}^{(-1)})$. By Proposition 2.1 we have that

$$h_0\eta_k \in \mathfrak{L}^2(E, G).$$

Then

$$T_k(h_0\eta_k) = h_0(t)g_{kk}^{(-1)}(t)\mathbf{e}_k(t) = \varphi_0(t)\mathbf{e}_k(t).$$

By Proposition 2.1 it follows that $\varphi_0(\cdot)\mathbf{e}_k(\cdot) \notin \mathfrak{L}^2(E, G)$. Hence T_k is not a bounded projection on $\mathfrak{L}^2(E, G)$. $\square$

Theorem 2.1. *The transformation T_k is a bounded projection in $\mathfrak{L}^2(E, G)$ if and only if*

$$g_{kk}g_{kk}^{(-1)} \in L^\infty(E). \tag{12}$$

Proof. We begin with proving that (12) is a necessary condition. The proof goes by contradiction. Suppose that $g_{kk}g_{kk}^{(-1)} \notin L^\infty(E)$. Then there exists a function $\psi_1 \in L^2(E)$ such that the function $\psi_1 g_{kk}g_{kk}^{(-1)} \notin L^2(E)$. Hence, $\psi_2 := \psi_1\sqrt{g_{kk}}g_{kk}^{(-1)} \notin L^2(E, g_{kk})$. We write

$$\int_E |\psi_2(t)|^2 \frac{1}{g_{kk}^{(-1)}(t)} dt = \int_E |\psi_1(t)|^2 g_{kk}(t)g_{kk}^{(-1)}(t)dt.$$

If the last integral is finite then by Lemma 2.2 with $\varphi_0 = \psi_2$ we obtain that the transformation T_k is not a bounded projection in $\mathfrak{L}^2(E, G)$. If the last integral is infinite then setting $\varphi_0 := \psi_1\sqrt{g_{kk}^{(-1)}}$ we get on the one hand that $\varphi_0 \notin L^2(E, g_{kk})$. On the other hand we have that

$$\int_E |\varphi_0(t)|^2 \frac{1}{g_{kk}^{(-1)}(t)} dt = \int_E |\psi_1(t)|^2 dt < +\infty$$

and by Lemma 2.2 it follows that the transformation T_k is not a bounded projection in $\mathfrak{L}^2(E, G)$. Thus if T_k is a bounded projection in $\mathfrak{L}^2(E, G)$ then (12) holds.

Let us prove the sufficiency of condition (12). Suppose $g_{kk}g_{kk}^{(-1)} \le C^2$ almost everywhere. Let $f = (f_1, \dots, f_n) \in \mathfrak{L}^2(E, G)$. Take any function $\psi \in L^2(E, g_{kk})$. Then $\psi g_{kk} \in L^2(E, g_{kk}^{(-1)})$ and

$$\|\psi g_{kk}\|_{L^2(E, g_{kk}^{(-1)})} \le C\|\psi\|_{L^2(E, g_{kk})}.$$

Using Proposition 2.1 we obtain

$$\int_E f_k(t)\psi(t)g_{kk}(t)dt = \langle f|\bar{\psi} g_{kk}\eta_k\rangle_G \le \|f\|_{\mathfrak{L}^2(E,G)}\|\bar{\psi} g_{kk}\eta_k\|_{\mathfrak{L}^2(E,G)}$$

$$= \|f\|_{\mathfrak{L}^2(E,G)}\|\psi g_{kk}\|_{L^2(E, g_{kk}^{(-1)})} \le C\|f\|_{\mathfrak{L}^2(E,G)}\|\psi\|_{L^2(E, g_{kk})}.$$

This implies

$$\|f_k\|_{L^2(E, g_{kk})} \le C\|f\|_{\mathfrak{L}^2(E,G)}$$

and completes the proof.

□

The following example shows that there exist HPD matrices for which the condition (12) does not hold.

Example 2.1. *Let $\alpha > 0$ and $t \in (0, 1)$ then the following 2×2 matrices*

$$G_\alpha(t) = \begin{pmatrix} t^{-\alpha} & \sqrt{t^{-2\alpha} - 1} \\ \sqrt{t^{-2\alpha} - 1} & t^{-\alpha} \end{pmatrix}$$

are HPD matrices for any $t \in (0, 1)$ and the condition (12) does not hold.

Proof. It is easy to check that for any $a \in (0, 1)$ the matrix

$$\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

is HPD. We have that $\det G_\alpha(t) \equiv 1$ for $t \in (0,1)$. Hence,

$$G_\alpha^{-1}(t) = \begin{pmatrix} t^{-\alpha} & -\sqrt{t^{-2\alpha}-1} \\ -\sqrt{t^{-2\alpha}-1} & t^{-\alpha} \end{pmatrix}.$$

□

By Theorem 2.1 and Proposition 2.1 we obtain

Proposition 2.2. *If the condition (12) holds for all k $(1 \le k \le n)$ then there exists $C_0 > 1$ such that for any $f = \langle f_1, f_2, \ldots, f_n \rangle \in \mathfrak{L}^2(E, G)$*

$$\|f\|_{\mathfrak{L}^2(E,G)} \le \sum_{k=1}^{n} \|f_k\|_{L^2(E,g_{kk})} \le C_0 \|f\|_{\mathfrak{L}^2(E,G)}.$$

Proof. We write $f(t) = \sum_{k=1}^n f_k(t)\mathbf{e}_k(t)$ and afterwards apply the triangle inequality and Proposition 2.1 to prove the left hand inequality. The right hand inequality follows directly from Theorem 2.1. □

Consider the following measurable basis of V

$$\left\{ \frac{1}{\sqrt{g_{11}(t)}}\mathbf{e}_1(t), \eta_2(t), \eta_3(t), \ldots, \eta_n(t) \right\}. \tag{13}$$

For any $t \in E$ the matrix of our Hermitian form with respect to the basis (13) has the following form

$$G'(t) = A^*(t) \cdot G(t) \cdot A(t), \tag{14}$$

where

$$A(t) = \begin{pmatrix} \frac{1}{\sqrt{g_{11}(t)}} & g_{12}^{(-1)}(t) & \cdot\ \cdot\ \cdot & g_{1n}^{(-1)}(t) \\ 0 & g_{22}^{(-1)}(t) & \cdot\ \cdot\ \cdot & g_{2n}^{(-1)}(t) \\ 0 & g_{32}^{(-1)}(t) & \cdot\ \cdot\ \cdot & g_{3n}^{(-1)}(t) \\ 0 & & \cdot\ \cdot\ \cdot & \\ 0 & & \cdot\ \cdot\ \cdot & \\ 0 & g_{n2}^{(-1)}(t) & \cdot\ \cdot\ \cdot & g_{nn}^{(-1)}(t) \end{pmatrix};$$

$$A^*(t) = \begin{pmatrix} \frac{1}{\sqrt{g_{11}(t)}} & 0 & 0 & 0 & 0 & 0 \\ \overline{g_{12}^{(-1)}(t)} & \overline{g_{22}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{n2}^{(-1)}(t)} \\ \overline{g_{13}^{(-1)}(t)} & \overline{g_{23}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{n3}^{(-1)}(t)} \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \overline{g_{1n}^{(-1)}(t)} & \overline{g_{2n}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{nn}^{(-1)}(t)} \end{pmatrix}.$$

Matrix multiplication operations yield

$$G'(t) = \begin{pmatrix} \chi_E(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{g_{22}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{n2}^{(-1)}(t)} \\ 0 & \overline{g_{23}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{n3}^{(-1)}(t)} \\ 0 & & \cdot & \cdot & \cdot & \\ 0 & & \cdot & \cdot & \cdot & \\ 0 & \overline{g_{2n}^{(-1)}(t)} & \cdot & \cdot & \cdot & \overline{g_{nn}^{(-1)}(t)} \end{pmatrix}.$$

By Proposition 2.1 we have that $\frac{1}{\sqrt{g_{11}(t)}}\mathbf{e}_1(t) \in \mathfrak{L}^2(E,G)$. For any $t \in E$ we consider the subspace $V_0(t)$ of V generated by linearly independent vectors $\{\zeta_j(t)\}_{j=2}^n$. We have that

$$\overline{g_{kj}^{(-1)}(t)} = G(\eta_j(t), \eta_k(t)) := h_{j-1k-1}(t) \qquad 2 \le j,k \le n.$$

Hence

$$H(t) = \Big(h_{jk}(t)\Big)_{1 \le j,k \le n-1}$$

is a HPD matrix. The equation (14) yields

$$\det H(t) = \frac{g_{11}(t)}{\det G(t)}. \tag{15}$$

By induction we obtain the following

Theorem 2.2. *There exist a measurable basis $\{\varepsilon_j(t)\}_{j=1}^n$ of V such that $\varepsilon_j \in \mathfrak{L}^2(E,G), 1 \le j \le n$ and the matrix $\mathfrak{E}(t)$ of the Hermitian form with* *respect to the basis $\{\varepsilon_j(t)\}_{j=1}^n$ has the following form*

$$\mathfrak{E}(t) = \begin{pmatrix} \chi_E(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & \chi_E(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & \chi_E(t) & 0 & 0 & 0 \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 0 & 0 & 0 & 0 & 0 & \chi_E(t) \end{pmatrix}.$$

3. Some classical systems in $\mathfrak{L}^2(E, G)$

Let $\Phi = \{\varphi_m\}_{m=1}^{\infty}$ be a complete orthonormal system defined on a measurable set $E \subset \mathbb{R}$. We will study the matrices (3) for which the system $\{\varphi_m(t)\mathbf{e}_j,\ 1 \le j \le n; m \in \mathbb{Z}\}$ is a basis in some sense in the spaces $\mathfrak{L}^2(E, G)$. It will be natural to begin our study with the Haar and the trigonometric systems. In this section we prove some preliminary results which can be used for both systems. A system of functions $\{\phi_m\}_{m=1}^{\infty} \subset L^{\infty}(E)$ is called total with respect to $L(E)$ if

$$\int_E g(t)\phi_m(t)dt = 0 \quad \text{for some} \quad g \in L(E) \quad \text{and for all}\, m \in \mathbb{N}$$

if and only if $g(t) = 0$ a.e.

Lemma 3.1. *Let $\Phi = \{\varphi_m\}_{m=1}^{\infty} \subset L^{\infty}(E)$ be an orthonormal system defined on a measurable set $E \subset \mathbb{R}$ total with respect to $L(E)$. Let $G(t)$ be an HPD matrix for a.e. $t \in E$ such that $g_{jj} \in L(E)$ for all $1 \le j \le n$. Then the system $\{\varphi_m(t)\mathbf{e}_j,\ 1 \le j \le n; m \in \mathbb{Z}\}$ is complete in $\mathfrak{L}^2(E, G)$.*

Proof. Suppose that for some $f \in \mathfrak{L}^2(E, G)$

$$\langle \varphi_m(t)\mathbf{e}_j | f\rangle_G = 0, \qquad 1 \le j \le n; \quad m \in \mathbb{N}. \tag{16}$$

For a fixed j $(1 \le j \le n)$ we have that

$$f^*(t)G(t)\varphi_m(t)\mathbf{e}_j = \varphi_m(t)\sum_{k=1}^n g_{jk}(t)\overline{f_k(t)}.$$

Hence, by (16) it follows

$$\int_{\mathbb{T}} \varphi_m(t)\sum_{k=1}^n \overline{f_k(t)}g_{jk}(t)dt = 0, \qquad 1 \le j \le n; \quad m \in \mathbb{N}.$$

Then our assumption that Φ is total with respect to $L(E)$ implies

$$\sum_{k=1}^{n} \overline{f_k(t)} g_{jk}(t) = 0 \quad \text{a.e. on } E, \quad 1 \le j \le n.$$

Recall that $\det G(t) \neq 0$ a.e. on E. Thus $f_k(t) = 0$ a.e. on E for all $1 \le k \le n$. $\square$

3.1. The Haar system in $\mathfrak{L}^2([0,1], G)$

Let $\mathcal{H} = \{h_k\}_{k=0}^{\infty}$ be the Haar system enumerated in its natural order. Let Δ be a collection of all dyadic intervals of $[0,1]$

$$\Delta := \{I \subset [0,1] : I = [(l-1)2^{-n}, l2^{-n}), n = 0,1,\ldots, l = 1,\ldots,2^n\} \cup \{[0,1]\}$$

and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. Let w be a nonnegative integrable function on $[0,1]$. We say that w belongs to Muckenhoupt's dyadic class $A_p^{(d)}$, $1 \le p < \infty$ if there exists a constant $B(p) > 0$ such that for any $I \in \Delta$ we have

$$\int_I w(t)dt \left[\int_I w(t)^{-\frac{1}{p-1}} dt \right]^{p-1} \le B(p)|I|^p. \tag{17}$$

A.S. Krantzberg [8] proved that the condition $w \in A_p^{(d)}$ is a necessary and sufficient condition for the Haar system $\mathcal{H}$ to be a basis of $L^p([0,1], w)$, $1 < p < \infty$. In [4] it was pointed out that the condition $w \in A_p^{(d)}$ guarantees that $\mathcal{H}$ is an unconditional basis of $L^p([0,1], w)$, $1 < p < \infty$. In [5] we prove that the condition $w \in A_p^{(d)}$ implies that $\mathcal{H}$ is a greedy basis of $L^p([0,1], w)$, $1 < p < \infty$.

In this subsection we consider $V = \mathbb{R}^n$, $\mathbb{K} = \mathbb{R}$ and the domain of the vector-valued functions will be $[0,1]$. Furthermore, $\{\mathbf{e}_j\}_{j=1}^n$ will denote the standard basis for the space $\mathbb{R}^n$.

For any $t \in [0,1]$ let $G_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear form such that the functions

$$g_{jk}(t) = G_t(\mathbf{e}_j, \mathbf{e}_k), \qquad 1 \le j,k \le n,$$

are measurable and the matrix

$$G(t) = \Big(g_{jk}(t) \Big)_{1 \le j,k \le n}$$

is a symmetric positive definite matrix a.e. on $[0,1]$. Moreover, we suppose that $g_{jj} \in L^1([0,1])$ for all $1 \le j \le n$. By Lemma 3.1 it follows that the system

$$\mathfrak{H} = \{h_m(t)\mathbf{e}_j,\, 1 \le j \le n; m \in \mathbb{N}_0\}$$

is complete in $\mathfrak{L}^2([0,1],G)$.

Proposition 3.1. *The system $\mathfrak{H}$ is minimal in $\mathfrak{L}^2([0,1],G)$ if and only if*

$$g_{kk}^{(-1)} \in L([0,1]), \qquad 1 \le k \le n. \tag{18}$$

Proof. First, we prove sufficiency. Suppose that (18) holds. Then by Proposition 2.1

$$\mathfrak{H}^* = \{h_m(t)\eta_k(t),\, 1 \le k \le n; m \in \mathbb{N}_0\} \subset \mathfrak{L}^2([0,1],G). \tag{19}$$

We have that for any $1 \le k, j \le n$

$$\eta_k^*(t)G(t)\mathbf{e}_j = \sum_{l=1}^{n} g_{kl}^{(-1)}(t)g_{lj}(t) = \delta_{kj} \quad \text{a.e. on } [0,1].$$

This implies that for all $1 \le k, j \le n$

$$\langle h_l(t)\mathbf{e}_j | h_m(t)\eta_k\rangle_G = \delta_{kj}\delta_{lm}, \quad m, l \in \mathbb{N}_0. \tag{20}$$

Hence, $\mathfrak{H}$ is minimal in $\mathfrak{L}^2([0,1],G)$.

Second, we prove necessity. Suppose that $\mathfrak{H}$ is minimal in $\mathfrak{L}^2([0,1],G)$. Then for any fixed j_0 $(1 \le j_0 \le n)$ and m_0 $(m_0 \in \mathbb{N}_0)$ there exists $\zeta_{j_0m_0} \in \mathfrak{L}^2([0,1],G)$ such that for any $m \in \mathbb{N}_0$

$$\langle h_m(t)\mathbf{e}_j | \zeta_{j_0m_0}(t)\rangle_G = 0 \quad \text{for any } j \ne j_0 \quad (1 \le j \le n); \tag{21}$$

for any j $(1 \le j \le n)$

$$\langle h_m(t)\mathbf{e}_j | \zeta_{j_0m_0}(t)\rangle_G = 0 \quad \text{for any } m \ne m_0 \quad (m \in \mathbb{Z}); \tag{22}$$

$$\langle h_{m_0}(t)\mathbf{e}_{j_0} | \zeta_{j_0m_0}(t)\rangle_G = 1. \tag{23}$$

Let $\zeta_{j_0m_0}(t) = \langle \tau_1(t), \tau_2(t), \dots, \tau_n(t)\rangle$. Then by (21) we have that

$$\sum_{k=1}^{n} g_{jk}(t)\tau_k(t) = 0 \qquad \text{a.e. on } [0,1] \quad \text{for any } j \ne j_0 \quad (1 \le j \le n).$$

Hence, $\zeta_{j_0m_0}(t) = \varphi_0(t)\langle g_{j_01}^{(-1)}(t), g_{j_02}^{(-1)}(t), \dots, g_{j_0n}^{(-1)}(t)\rangle$. Which by (22) yields $\varphi_0(t) = c_0 h_{m_0}(t)$, where $c_0 \in \mathbb{R}$. By (23) it follows that $c_0 = 1$. Thus by Proposition 2.1 with $\psi = h_0$ we obtain (18). $\square$

By Lemma 3.1 we obtain

Proposition 3.2. *The system $\mathfrak{H}$ is complete in $\mathfrak{L}^2([0,1],G)$.*

For any fixed j $(1 \le j \le n)$ consider the partial sums

$$S_{Nj}(f,\mathfrak{H},t) = \mathbf{e}_j \sum_{m=0}^{N} \langle f|h_m\eta_j\rangle_G \, h_m(t), \qquad N \in \mathbb{N}_0. \tag{24}$$

We have that

$$\eta_j^*(t)G(t)f(t) = \sum_{l=1}^{n}\sum_{k=1}^{n} g_{lj}^{(-1)}(t)g_{lk}(t)f_k(t) = f_j(t).$$

Thus we obtain that

$$S_{Nj}(f,\mathfrak{H},t) = \sum_{m=0}^{N} a_m(f_j)h_m(t), \qquad N \in \mathbb{N}_0, \tag{25}$$

where

$$a_m(\phi) = \int_{[0,1]} \phi(t)h_m(t)dt, \qquad m \in \mathbb{N}_0. \tag{26}$$

We will say that the system $\mathfrak{H}$ has natural enumeration if for any j $(1 \le j \le n)$ it maintains the order of enumeration between the elements of the system $\{h_m\mathbf{e}_j\}_{m\in\mathbb{N}_0}$.

Theorem 3.1. *Let the system $\mathfrak{H}$ be given in a natural enumeration and suppose that for all k $(1 \le k \le n)$ (12), (18) hold. Then the system $\mathfrak{H}$ is a Schauder basis in $\mathfrak{L}^2([0,1],G)$ if and only if there exists $C>1$ such that for any dyadic interval $I \subset [0,1]$.*

$$\int_I g_{kk}(t)dt \int_I \frac{1}{g_{kk}(t)}dt \le C|I|^2 \qquad \textit{for all} \quad k \quad (1 \le k \le n). \tag{27}$$

Proof. Sufficiency. Let $f = \langle f_1, f_2, \ldots, f_n\rangle \in \mathfrak{L}^2([0,1],G)$. By (24), (25) any partial sum with respect to the system $\mathfrak{H}$ of the function f has the following form

$$\sum_{j=1}^{n} S_{N_j j}(f,t) = \sum_{j=1}^{n} \mathbf{e}_j S_{N_j}(f_j,t),$$

where $S_{N_j}(f_j, t)$ is the N_jth partial sum of the Fourier series of the function f_j. It is well known (see [8], [4]) that if (27) holds then we will have that for some $C_3 > 1$ and for any j $(1 \leq j \leq n)$

$$\|S_{N_j}(f_j, \cdot)\|_{L^2([0,1],g_{jj})} \leq C_3 \|f_j\|_{L^2([0,1],g_{jj})}.$$

Hence, by Proposition 2.2 the proof of sufficiency is finished.

Necessity. By Proposition 2.2 $\mathfrak{H}$ is a Schauder basis in $\mathfrak{L}^2([0,1], G)$ if and only if the above inequalities hold for all j $(1 \leq j \leq n)$. Hence, as in [8], [4] we finish the proof. □

Theorem 3.2. *Suppose that for for all k $(1 \leq k \leq n)$ (12), (18) hold. Then the system $\mathfrak{H}$ is an unconditional basis in $\mathfrak{L}^2([0,1], G)$ if and only if there exists $C > 1$ such that for any dyadic interval $I \subset [0,1]$ (27) holds.*

Proof. Evidently we have to check only the sufficiency. Let $\mathfrak{H}_\sigma$ be the system $\mathfrak{H}$ enumerated in any arbitrary order. Then for any $f = \langle f_1, f_2, \ldots, f_n \rangle \in \mathfrak{L}^2([0,1], G)$ by (24), (25) we will have that any partial sum with respect to the system $\mathfrak{H}_\sigma$ of the function f has the following form

$$\sum_{j=1}^{n} S_{N_{\sigma(j)}j}(f, t) = \sum_{j=1}^{n} \mathbf{e}_j S_{N_{\sigma(j)}}(f_j, t).$$

Afterwards we apply conditions for which the Haar system is an unconditional basis in a weighted norm space $L^p, 1 < p < \infty$ (see the beginning of this section) and finish the proof.

□

3.2. The trigonometric system in $\mathfrak{L}^2(\mathbb{T}, G)$

Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and let w be a nonnegative function defined on $\mathbb{T}$. We say that $w \in A_p$, $1 < p < \infty$ if for some $B(p) > 0$ and for any interval $I \subset \mathbb{T}$ (17) holds. The proof that the trigonometric system is a basis in any $L^p(\mathbb{T})$ $1 < p < \infty$ is based on the fact that the operator that maps a function into its trigonometrical conjugate function is a bounded operator from $L^p(\mathbb{T})$ into itself (see e.g. [15]). Hence, after the result by Hunt, Muckenhoupt and Wheeden [3] that the trigonometrical conjugate operator is a bounded operator from $L^p(\mathbb{T}, w)$ into itself if and only if $w \in A_p$ it was easy to observe that the trigonometric system is a basis in $L^p(\mathbb{T}, w)$ $1 < p < \infty$ if and only if $w \in A_p$ (see e.g. [4]).

In this section we specify $V = \mathbb{C}^n$, $\mathbb{K} = \mathbb{C}$ and the domain of the vector-valued function will be $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Furthermore, $\{\mathbf{e}_j\}_{j=1}^n$ will denote the standard basis for the space $\mathbb{C}^n$

For any $t \in \mathbb{T}$ let $G_t : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a Hermitian form such that the functions

$$g_{jk}(t) = G_t(\mathbf{e}_j, \mathbf{e}_k), \qquad 1 \le j, k \le n,$$

are measurable and the matrix

$$G(t) = \Big(g_{jk}(t) \Big)_{1 \le j,k \le n}$$

is a HPD matrix a.e. on $\mathbb{T}$. Moreover, we suppose that $g_{jj} \in L^1(\mathbb{T})$ for all $1 \le j \le n$.

We consider the following system

$$\mathfrak{T} = \{e^{i2\pi mt}\mathbf{e}_j,\ 1 \le j \le n; m \in \mathbb{Z}\} \tag{28}$$

in $\mathfrak{L}^2(\mathbb{T}, G)$. As a corollary of Lemma 3.1 we obtain

Proposition 3.3. *The system $\mathfrak{T}$ is complete in $\mathfrak{L}^2(\mathbb{T}, G)$.*

As in the case of the Haar system we obtain.

Proposition 3.4. *The system $\mathfrak{T}$ is minimal in $\mathfrak{L}^2(\mathbb{T}, G)$ if and only if*

$$g_{jj}^{(-1)} \in L^1(\mathbb{T}), \qquad 1 \le j \le n. \tag{29}$$

For any fixed j $(1 \le j \le n)$ consider the partial sums

$$S_{Nj}(f, t) = \mathbf{e}_j \sum_{|m| \le N} \langle f | e^{i2\pi m \cdot} \eta_j \rangle_G \, e^{i2\pi mt}, \qquad N = 1, 2, \ldots. \tag{30}$$

We have that

$$\eta_j^*(t)G(t)f(t) = \sum_{l=1}^{n} \sum_{k=1}^{n} \overline{g_{lj}^{(-1)}}(t) g_{lk}(t) f_k(t) = f_j(t).$$

Thus we obtain that

$$S_{Nj}(f, t) = \mathbf{e}_j \sum_{|m| \le N} c_m(f_j) e^{i2\pi mt}, \qquad N = 1, 2, \ldots, \tag{31}$$

where

$$c_m(\phi) = \int_{\mathbb{T}} \phi(t) e^{-i2\pi mt} dt. \tag{32}$$

Theorem 3.3. *Suppose that for all k $(1 \le k \le n)$*

$$g_{kk}\, g_{kk}^{(-1)} \in L^{\infty}(\mathbb{T}) \tag{33}$$

and that the conditions (29) hold for all j $(1 \le j \le n)$. Then the system $\mathfrak{T}$ is a Riesz basis in $\mathfrak{L}^2(\mathbb{T}, G)$ if and only if there exists $C_1 > 1$ such that

$$C_1^{-1} \le g_{kk}(t) \le C_1 \qquad a.e. \text{ on } \mathbb{T} \quad \text{for all} \quad k \quad (1 \le k \le n). \tag{34}$$

Proof. Sufficiency. Let $f = \langle f_1, f_2, \dots, f_n \rangle \in \mathfrak{L}^2(\mathbb{T}, G)$. Then by Theorem 2.1 and Proposition 2.1 we have that for all k $(1 \le k \le n)$ $f_k \in L^2(\mathbb{T}, g_{kk})$. Hence, by (34) it follows that $f_k \in L^2(\mathbb{T})$ for all k $(1 \le k \le n)$. This together with (31), (32) imply that the sequence of the coefficients of the expansion of f with respect to the system $\mathfrak{T}$ belongs to l^2. Now let us show that for any j $(1 \le j \le n)$ the series $\mathbf{e}_j \sum_{m=-\infty}^{+\infty} b_m e^{i2\pi mt}$ converges in $\mathfrak{L}^2(\mathbb{T}, G)$ for any $\{b_m\} \in l^2$. By Proposition 2.1 it is equivalent to the convergence of the series $\sum_{m=-\infty}^{+\infty} b_m e^{i2\pi mt}$ in the space $L^2(\mathbb{T}, g_{jj})$. By (34) we have that the norms in $L^2(\mathbb{T}, g_{jj})$ and $L^2(\mathbb{T})$ are equivalent, which finishes the proof.

Necessity. Suppose that for some k_0 $(1 \le k_0 \le n)$

$$g_{k_0 k_0} \notin L^{\infty}(\mathbb{T}) \qquad \text{or} \qquad \frac{1}{g_{k_0 k_0}} \notin L^{\infty}(\mathbb{T}).$$

Let us consider the first case. There exists $\phi_0 \in L^2(\mathbb{T})$ such that $\sqrt{g_{k_0 k_0}}\phi_0 \notin L^2(\mathbb{T})$. Evidently $\{c_m(\phi_0)\} \in l^2$ but the series $\sqrt{g_{k_0 k_0}(t)} \sum_{m=-\infty}^{+\infty} c_m(\phi_0) e^{i2\pi mt}$ diverges in $L^2(\mathbb{T})$. Which leads to a contradiction. $\square$

Let $\rho : \mathbb{N}_0 \to \mathbb{Z}$ be a bijection defined by the following relations

$$\rho(k) = \begin{cases} -\frac{k+1}{2} & \text{if} \quad k = 2l - 1 \\ \frac{k}{2} & \text{if} \quad k = 2l; \quad l = 0, 1, 2, \dots . \end{cases}$$

We will say that the set $\mathbb{Z}$ has natural enumeration if it is enumerated by means of the inverse of ρ. The corresponding enumeration of the trigonometric system $\{e^{i2\pi mt}\}_{m \in \mathbb{Z}}$ is also called natural enumeration. We will say that the order of enumeration of the system $\mathfrak{T}$ is natural if it maintains the order of natural enumeration between the elements of the system $\{e^{i2\pi mt}\mathbf{e}_j\}_{m \in \mathbb{Z}}$ for any j $(1 \le j \le n)$.

Theorem 3.4. *Let the system $\mathfrak{T}$ be given with a natural enumeration and suppose that for all k, j $(1 \le k, j \le n)$ the conditions (33), (29) hold. Then the system $\mathfrak{T}$ is a Schauder basis in $\mathfrak{L}^2(\mathbb{T}, G)$ if and only if there exists $C_2 > 1$ such that for any interval $I \subset \mathbb{T}$*

$$\int_I g_{kk}(t)dt \int_I \frac{1}{g_{kk}(t)}dt \le C_2|I|^2 \qquad \textit{for all} \quad k \quad (1 \le k \le n). \tag{35}$$

Proof. Sufficiency. Let $f = \langle f_1, f_2, \dots, f_n\rangle \in \mathfrak{L}^2(\mathbb{T}, G)$. By (30), (31) any partial sum with respect to the system $\mathfrak{T}$ of the function f has the following form

$$\sum_{j=1}^{n} S_{N_j j}(f, t) = \sum_{j=1}^{n} \mathbf{e}_j S_{N_j}(f_j, t),$$

where $S_{N_j}(f_j, t)$ is the N_jth partial sum of the Fourier series of the function f_j. We will have that for some $C_3 > 1$ for any j $(1 \le j \le n)$

$$\|S_{N_j}(f_j, \cdot)\|_{L^2(\mathbb{T}, g_{jj})} \le C_3\|f_j\|_{L^2(\mathbb{T}, g_{jj})}.$$

Hence, by Proposition 2.2 the proof of sufficiency is finished.

Necessity. By Proposition 2.2 $\mathfrak{T}$ is a Schauder basis in $\mathfrak{L}^2(\mathbb{T}, G)$ if and only if the above inequalities hold for all j $(1 \le j \le n)$. The proof of the remaining part one can obtain repeating by small modifications of the proof in [3] or directly applying the result obtained in [10] where necessary and sufficient conditions on the weight function w were found such that the trigonometric system is a basis in $L^p(\mathbb{T}, w)$ with respect to Abel-Poisson summation method. In this case also the weight function w should belong to the class A_p. $\square$

4. Simultaneous greedy approximation

In this section we are going to discuss simultaneous approximation of n elements $F_1, \dots, F_n$ of some Banach spaces $X_1, \dots, X_n$ with respect to a system Ψ which is a basis in each of those spaces. We interpret this problem as a problem of approximation of a single element $(F_1, \dots, F_n)$ in a new Banach space with respect to a new dictionary. Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^{\infty}$ be a basis for X. For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f, \Psi)_X := \inf_{b_k, \Lambda} \|f - \sum_{k \in \Lambda} b_k\psi_k\|_X,$$

where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j, j = 1, 2, ...,$ of the positive integers decreasing and write $\rho \in D(f)$ if

$$|c_{k_1}(f)|\|\psi_{k_1}\|_X \geq |c_{k_2}(f)|\|\psi_{k_2}\|_X \geq ... \,.$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f) := G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme for m-term approximation of an element f. This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA). The best we can achieve with the algorithm G_m is

$$\|f - G_m(f)\|_X = \sigma_m(f, \Psi)_X,$$

or a little weaker

$$\|f - G_m(f)\|_X \leq C\sigma_m(f, \Psi)_X$$

for all $f \in X$ with a constant $C > 0$ independent of f and m. The following concept of a greedy basis was introduced in [6].

Definition 4.1. *We call a basis Ψ a greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f, \Psi, \rho)\|_X \leq C\sigma_m(f, \Psi)_X$$

with a constant C independent of f and m.

The reader can find a discussion of greedy bases in [7], [13], [14], and [12], Chapter 1. The following characterization theorem was proved in [6].

Theorem 4.1. *A basis is greedy if and only if it is unconditional and democratic.*

Definition 4.2. *We say that a system $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a democratic system for X if there exists a constant $D := D(X, \Psi)$ such that, for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$, we have*

$$\left\| \sum_{k \in P} \frac{\psi_k}{\|\psi_k\|_X} \right\| \le D \left\| \sum_{k \in Q} \frac{\psi_k}{\|\psi_k\|_X} \right\|. \tag{36}$$

Given some Banach spaces $X_1, \dots, X_n$, we consider a new Banach space

$$X^n := \{F = (F_1, \dots, F_n),\ F_j \in X,\ j = 1, \dots, n\}.$$

Denote $F\mathbf{e}_j := (0, \dots, 0, F_j, 0, \dots, 0)$. Then X^n has the properties
i) $\|F\mathbf{e}_j\|_{X^n} = \|F_j\|_{X_j}$ for all $j, 1 \le j \le n$;
ii) there exist $0 < c \le C < \infty$ such that

$$c \left(\sum_{j=1}^{n} \|F_j\|_{X_j}^2 \right)^{1/2} \le \|F\|_{X^n} \le C \left(\sum_{j=1}^{n} \|F_j\|_{X_j}^2 \right)^{1/2}. \tag{37}$$

Dictionary Ψ generates a dictionary Ψ^n in the space X^n defined as follows

$$\Psi^n := \{\psi_k \mathbf{e}_j = (0, \dots, 0, \psi_n, 0, \dots, 0),\ \psi_k \in \Psi,\ j = 1, \dots, n\}.$$

It is clear that if Ψ is an unconditional basis of all $X_j, 1 \le j \le n$ if and only if Ψ^n is an unconditional basis of X^n.

For any j, $1 \le j \le n$ define

$$\varphi_j(\ell) := \sup_{A:|A| \le \ell} \left\| \sum_{\nu \in A} \frac{\psi_\nu}{\|\psi_\nu\|_{X_j}} \right\|_{X_j}.$$

It is clear from this definition that

$$\varphi_j(m + l) \le \varphi_j(m) + \varphi_j(l). \tag{38}$$

For any $A \subset \mathbb{N}$ with $|A| = \ell$ we have

$$D_j^{-1} \varphi_j(\ell) \le \left\| \sum_{k \in A} \frac{\psi_k}{\|\psi_k\|_{X_j}} \right\|_{X_j} \le \varphi_j(\ell), \qquad 1 \le j \le n. \tag{39}$$

Let

$$\varphi(\ell) := \max_{1 \le j \le n} \varphi_j(\ell) \qquad \text{and} \qquad \eta(\ell) := \min_{1 \le j \le n} \varphi_j(\ell). \tag{40}$$

Definition 4.3. *We say that a system Ψ is equidemocratic in the spaces $X_j, 1 \le j \le n$ if Ψ is democratic in each space $X_j, 1 \le j \le n$ and for some $c_1 > 0$*

$$c_1\varphi(\ell) \le \eta(\ell) \le \varphi(\ell) \qquad \textit{for all} \quad \ell \in \mathbb{N}.$$

The following result holds.

Theorem 4.2. *Let Ψ be a system which belongs to all spaces X_j, $1 \le j \le n$. Then Ψ^n is a democratic system of X^n if and only if Ψ is a equidemocratic system for the spaces X_j, $1 \le j \le n$.*

Proof. Suppose that Ψ is a democratic system of X^d. We fix any j, ν $(1 \le j, \nu \le n)$ and for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have

$$\Big\| \sum_{k \in P} \frac{\psi_k \mathbf{e}_j}{\|\psi_k \mathbf{e}_j\|_{X^n}} \Big\|_{X^n} \le D \Big\| \sum_{k \in Q} \frac{\psi_k \mathbf{e}_\nu}{\|\psi_k \mathbf{e}_\nu\|_{X^n}} \Big\|_{X^n}.$$

Using the property i) of the space X^n we obtain that

$$\Big\| \sum_{k \in P} \frac{\psi_k}{\|\psi_k\|_{X_j}} \Big\|_{X_j} \le D \Big\| \sum_{k \in Q} \frac{\psi_k}{\|\psi_k\|_{X_\nu}} \Big\|_{X_\nu}.$$

Hence, the system Ψ is equidemocratic for the spaces $X_j, 1 \le j \le n$.

Sufficiency. Let $B = \{(k, \nu) : k \in \mathbb{N}, 1 \le \nu \le n\}$ be of cardinality $|B| = \ell$. Denote $B_j := \{k : (k, j) \in B\}$. Then on one hand we have

$$\begin{aligned} \Big\| \sum_{(k,j) \in B} \frac{\psi_k e_j}{\|\psi_k e_j\|_{X^n}} \Big\|_{X^n} &\le C \bigg(\sum_{j=1}^{n} \Big\| \sum_{k \in B_j} \frac{\psi_k}{\|\psi_k\|_{X_j}} \Big\|_{X_j}^2 \bigg)^{1/2} \\ &\le C \bigg(\sum_{j=1}^{n} \varphi_j(|B_j|)^2 \bigg)^{1/2} \le C n^{1/2} \varphi(\ell). \end{aligned} \tag{41}$$

On the other hand at least for some $\nu, 1 \le \nu \le n$ we have $|B_\nu| \ge s(\ell, n)$, where $s(\ell, n) = \ell/n$ if ℓ/n is an integer and $s(\ell, n) = [ell/n] + 1$ otherwise. Therefore

$$\Big\| \sum_{(k,j) \in B} \frac{\psi_k \mathbf{e}_j}{\|\psi_k \mathbf{e}_j\|_{X^n}} \Big\|_{X^n} \ge c \bigg(\sum_{j=1}^{n} \Big\| \sum_{k \in B_j} \frac{\psi_k}{\|\psi_k\|_{X_j}} \Big\|_{X_j}^2 \bigg)^{1/2} \ge c D_\nu^{-1} \varphi_\nu(s(\ell, n)),$$

where the last inequality follows by (39). Using the inequality (38) we obtain $\varphi_\nu(s(\ell,n)) \geq n^{-1}\varphi_\nu(\ell)$. Thus, by (41) we obtain that Ψ^n is democratic for X^n. $\square$

Theorem 4.3. *Ψ^n is a greedy basis of X^n if and only if Ψ is a greedy basis of the spaces $X_j, 1 \leq j \leq n$ and Ψ is equidemocratic for the spaces $X_j, 1 \leq j \leq n$.*

Proof. By Theorem 4.1 a basis is greedy if and only if it is unconditional and democratic. As it was mentioned above Ψ^n is an unconditional basis of X^n if and only if Ψ is an unconditional basis of the spaces $X_j, 1 \leq j \leq n$. Hence by Theorem 4.2 the proof is complete. $\square$

Propositions 2.1 and 2.2 show when one can apply Theorem 4.3 for the spaces $\mathfrak{L}^2(E,G)$.

By Theorems 4.3 and 3.2 we obtain

Theorem 4.4. *Suppose that for all k $(1 \leq k \leq n)$ (12), (18) hold. Then the system $\mathfrak{H}$ is a greedy basis in $\mathfrak{L}^2([0,1],G)$ if and only if there exists $C > 1$ such that for any dyadic interval $I \subset [0,1]$ (27) holds.*

Proof. We have to check only that the Haar system is equidemocratic for the spaces $L^2([0,1],g_{kk}), 1 \leq k \leq n$. For any $j, 1 \leq j \leq n$ define

$$\varphi_j^H(N) := \sup_{A:|A|\leq N} \left\| \sum_{k\in A} \frac{h_k}{\|h_k\|_{L^2(g_{jj})}} \right\|_{X_{L^2(g_{jj})}}.$$

In [5] it was shown that $\varphi_j^H(N) \leq C_j N^{1/2}$ if (27) holds. It is easy to observe that $\varphi_j^H(N) \geq N^{1/2}$. Thus the Haar system is equidemocratic for the spaces $L^2([0,1],g_{kk}), 1 \leq k \leq n$. $\square$

The following theorem is an immediate corollary of Theorems 4.3 and 3.3.

Theorem 4.5. *Suppose that for all k $(1 \leq k \leq n)$ (33) holds and that the conditions (29) hold for all j $(1 \leq j \leq n)$. Then the system $\mathfrak{T}$ is a greedy basis in $\mathfrak{L}^2(\mathbb{T},G)$ if and only if (34) holds.*

Acknowledgements. The research of the first author was supported in part by the research grant MTM 2010-15790. The research of the second author was supported by the National Science Foundation Grant DMS-0906260.

References

[1] C. de Boor, R. DeVore and A. Ron, The structure of finitely generated shift-invariant subspaces in $L_2(\mathbb{R}^d)$, Journal of Functional Analysis 119 (1994), pp. 37–78.

[2] R.A. Horn, Matrix analysis, Cambridge University Press, Cambridge, New-York, 2007.

[3] R.Hunt, B.Muckenhoupt, R.L.Wheeden, Weighted norm inequalities for the conjugate functions and Hilbert transform, Trans. Amer. Math. Soc., 176 (1973), pp. 227–251.

[4] K.S. Kazarian, On bases and unconditional bases in the spaces $L^p(d\mu)$, $1 \le p < \infty$, Studia Mathematica, **71** (1982), pp. 227–249.

[5] K.S. Kazarian, V.N. Temlyakov, Greedy bases in L^p-spaces, Proc. of the Steklov Institute of Math., **280** (2013).

[6] S. V. Konyagin, and V. N. Temlyakov, A remark on greedy approximation in Banach spaces, East. J. Approx.5 (1999), pp. 365-379.

[7] S.V. Konyagin and V.N. Temlyakov, Greedy Approximation with Regard to bases and General Minimal Systems, Serdica Math. J., **28** (2002) 305–328.

[8] A.S. Krantzberg, On the basisity of the Haar system in weighted spaces, Mosk. Inst. Elektr. Mat. **24**, (1971).

[9] M.Nielsen, Trigonometric bases for matrix weighted L^p-spaces, Journ. Mathem. Analysis and Appl., 371 (2010), pp. 784–792.

[10] M.Rosenblum, Summability of Fourier series in $L\,\mathrm{span}\,p(d\mu)$, Trans. Amer. Math. Soc., 105 (1962), pp. 32–42.

[11] V.N. Temlyakov, The best m-term approximation and greedy algorithms, Adv. Comp. Math., **8** (1998), pp. 249–265.

[12] V.N. Temlyakov, Greedy Approximation, Cambridge University Press, 2011.

[13] V. N. Temlyakov, Nonlinear method of approximation, Found. Compt. Math., **3** (2003), pp. 33-107.

[14] Temlyakov, V.N. Greedy approximation, Acta Numerica, **17** (2008), pp. 235–409.

[15] A. Zygmund, Trigonometric series, v. 1, Cambridge Univ. Press, 1959.