Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases*

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Abstract

We study greedy approximation with respect to quasi-greedy bases. For the L_p space, $1 < p < \infty$, $p \neq 2$, we prove that the error of the mth greedy approximation is bounded by the error of best m-term approximation multiplied by an extra factor of order $m^{|1/p-1/2|}$.

1 Introduction

We study the efficiency of greedy algorithms for m-term nonlinear approximation with regard to quasi-greedy bases. Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^{\infty}$ be a normalized basis for X ($\|\psi_k\| = 1, k \in \mathbb{N}$). All bases considered in our paper are assumed to be normalized. For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f, \Psi)_X := \inf_{b_k, \Lambda} \|f - \sum_{k \in \Lambda} b_k \psi_k\|_X,$$

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where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j, j = 1, 2, ...,$ of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \geq |c_{k_2}(f)| \geq ... \;.$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f) := G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme for m-term approximation of an element f. This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA). The best we can achieve with the algorithm G_m is

$$\|f - G_m(f)\|_X = \sigma_m(f, \Psi)_X,$$

or a little weaker

$$\|f - G_m(f)\|_X \leq C\sigma_m(f, \Psi)_X$$

for all $f \in X$ with a constant C independent of f and m. The following concept of a greedy basis has been introduced in [1].

Definition 1.1. *We call a basis Ψ a greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f, \Psi, \rho)\|_X \leq C\sigma_m(f, \Psi)_X$$

with a constant C independent of f and m.

We refer the reader to a survey [5] for further discussion of greedy type bases. In this paper we are interested in special inequalities – Lebesgue-type inequalities – for greedy approximation.

Lebesgue [3] proved the following inequality: for any 2π-periodic continuous function f we have

$$\|f - S_n(f)\|_\infty \leq (4 + \frac{4}{\pi^2} \ln n) E_n(f)_\infty, \tag{1.1}$$

where $S_n(f)$ is the nth partial sum of the Fourier series of f and $E_n(f)_\infty$ is the error of the best approximation of f by the trigonometric polynomials of order n in the uniform norm $\|\cdot\|_\infty$. The inequality (1.1) relates the error of a particular method (S_n) of approximation by the trigonometric polynomials of order n to the best-possible error $E_n(f)_\infty$ of approximation by the trigonometric polynomials of order n. By the Lebesgue-type inequality we mean an inequality that provides an upper estimate for the error of a particular method of approximation of f by elements of a special form, say, form $\mathcal{A}$, by the best-possible approximation of f by elements of the form $\mathcal{A}$. In the case of approximation with regard to bases (or minimal systems), the Lebesgue-type inequalities are known both in linear and in nonlinear settings (see surveys [2], [4] and [5]).

By the Definition 1.1 greedy bases are those for which we have ideal (up to a multiplicative constant) Lebesgue inequalities for greedy approximation. In this paper we concentrate on a wider class of bases than greedy bases – quasi-greedy bases. The concept of quasi-greedy basis was introduced in [1].

Definition 1.2. *The basis Ψ is called quasi-greedy if there exists some constant C such that*

$$\sup_m \|G_m(f, \Psi)\| \leq C\|f\|.$$

Subsequently, Wojtaszczyk [7] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{n\to\infty} G_n(f) = f.$$

The main result of this paper is the following Lebesgue-type inequality for greedy approximation with respect to a quasi-greedy basis in the L_p spaces.

Theorem 1.1. *Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have*

$$\|f - G_m(f, \Psi)\|_{L_p} \leq C(p, \Psi) m^{|1/2-1/p|} \sigma_m(f, \Psi)_{L_p}. \tag{1.2}$$

We note that inequality (1.2) is known (see [7]) in the case of unconditional bases Ψ. Theorem 1.1 was announced in [6]. Theorem 1.1 does not cover the case $p = 2$. It is mentioned in [7] that in the case $p = 2$ one has the following inequality

$$\|f - G_m(f, \Psi)\| \le C(\log m)\sigma_m(f, \Psi).$$

We do not know if the above inequality is sharp in the sense that an extra factor $\log m$ cannot be replaced by a slower growing factor. The reader can find further discussion of this problem in [6].

2 Lebesgue-type inequalities for TGA

In our study of quasi-greedy bases we need the following known Lemma 2.1 (see, for instance, [5], p. 269). It will be convenient to define the quasi-greedy constant K to be the least constant such that

$$\|G_m(f)\| \le K\|f\| \quad \text{and} \quad \|f - G_m(f)\| \le K\|f\|, \quad f \in X.$$

Lemma 2.1. *Suppose Ψ is a quasi-greedy basis with a quasi-greedy constant K. Then, for any numbers a_j and any finite set of indices P, we have*

$$(2K)^{-2} \min_{j\in P} |a_j| \|\sum_{j\in P} \psi_j\| \le \|\sum_{j\in P} a_j\psi_j\| \le 2K \max_{j\in P} |a_j| \|\sum_{j\in P} \psi_j\|.$$

We will use the notation

$$a_n(f) := |c_{k_n}(f)|$$

for the decreasing rearrangement of the coefficients of f. For a set of indices Λ we define the corresponding partial sum as follows

$$S_\Lambda(f) := \sum_{k\in\Lambda} c_k(f)\psi_k.$$

We will often use the following assumption: There exists an increasing function $v(m) := v(m, \Psi)$ such that for any two sets of indices A and B, $|A| = |B| = m$ we have

$$\|\sum_{k\in A} \psi_k\| \le v(m)\|\sum_{k\in B} \psi_k\|. \tag{2.1}$$

We begin with a theorem for a Banach space X. Later on we will specify this theorem for the L_p spaces.

Theorem 2.1. *Let Ψ be a quasi-greedy basis of X satisfying assumption (2.1) with the property: For any set of indices Λ*

$$\|S_\Lambda(f)\| \le w(|\Lambda|)\|f\|.$$

Then for each $f \in X$

$$\|f - G_m(f)\| \le (1 + 2w(m) + (2K)^3 v(m)w(m))\sigma_m(f).$$

Proof. Let, for a given $\epsilon > 0$, a polynomial

$$p_m(f) = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

satisfy the inequality

$$\|f - p_m(f)\| \le \sigma_m(f) + \epsilon. \tag{2.2}$$

Denote by Q the set of indices picked by the greedy algorithm after m iterations

$$G_m(f) = \sum_{k\in Q} c_k(f)\psi_k.$$

We use the representation

$$f - G_m(f) = f - S_Q(f) = f - S_P(f) + S_P(f) - S_Q(f). \tag{2.3}$$

First, we bound

$$\|f - S_P(f)\| = \|f - p_m(f) - S_P(f - p_m(f)\| \le (1 + w(m))\|f - p_m(f)\|. \tag{2.4}$$

Second, we write

$$\|S_P(f) - S_Q(f)\| = \|S_{P\setminus Q}(f) - S_{Q\setminus P}(f)\| \le \|S_{P\setminus Q}(f)\| + \|S_{Q\setminus P}(f)\|. \tag{2.5}$$

We begin with estimating the second term in the right side of (2.5)

$$\|S_{Q\setminus P}(f)\| = \|S_{Q\setminus P}(f - p_m(f))\| \le w(m)\|f - p_m(f)\|. \tag{2.6}$$

For the first term we have by Lemma 2.1

$$\|S_{P\setminus Q}(f)\| \le 2K \max_{k\in P\setminus Q} |c_k(f)|\|\sum_{k\in P\setminus Q} \psi_k\|$$

$$\leq 2K \min_{k\in Q\setminus P} |c_k(f)| v(m) \| \sum_{k\in Q\setminus P} \psi_k \| \leq (2K)^3 v(m) \|S_{Q\setminus P}(f)\|. \tag{2.7}$$

Combining (2.2) – (2.7) we obtain

$$\|f - G_m(f)\| \leq (1 + 2w(m) + (2K)^3 v(m) w(m)) \sigma_m(f).$$

□

We define the following expansional best m-term approximation of f with regard to Ψ (see [5], p. 269)

$$\tilde{\sigma}_m(f) := \tilde{\sigma}_m(f, \Psi) := \inf_{|\Lambda|=m} \|f - \sum_{k\in\Lambda} c_k(f)\psi_k\|.$$

It is clear that $\sigma_m(f) \leq \tilde{\sigma}_m(f)$. It is known that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f, \Psi) \leq C(\Psi, X)\sigma_m(f, \Psi).$$

Theorem 2.2. *Let Ψ be a quasi-greedy basis of X satisfying assumption (2.1). Then for each $f \in X$*

$$\|f - G_m(f)\| \leq C(\Psi, X) v(m) \tilde{\sigma}_m(f).$$

Proof. Let, for a given $\epsilon > 0$, a set of indices B be such that $|B| = m$ and

$$\|f - S_B(f)\| \leq \tilde{\sigma}_m(f) + \epsilon. \tag{2.8}$$

Let as above

$$G_m(f) = \sum_{k\in Q} c_k(f)\psi_k.$$

Then

$$\|f - G_m(f)\| \leq \|f - S_B(f)\| + \|S_{B\setminus Q}(f)\| + \|S_{Q\setminus B}(f)\|. \tag{2.9}$$

Our assumption that Ψ is quasi-greedy gives

$$\|S_{Q\setminus B}(f)\| = \|S_{Q\setminus B}(f - S_B(f))\|$$

$$= \|G_{|Q\setminus B|}(f - S_B(f))\| \leq K\|f - S_B(f)\|. \tag{2.10}$$

Combining (2.8) – (2.10) and using (2.7) we obtain

$$\|f - G_m(f)\| \leq (1 + K + 8K^4 v(m))\tilde{\sigma}_m(f).$$

□

We now proceed to a discussion of quasi-greedy bases in L_p spaces. We use the brief notation $\|\cdot\|_p := \|\cdot\|_{L_p}$. We will use the following theorem from [6]. We note that in the case $p=2$ Theorem 2.3 was proved in [7]

Theorem 2.3. *Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of the L_p space, $1 < p < \infty$. Then for each $f \in X$ we have*

$$C_1(p) \sup_n n^{1/p} a_n(f) \le \|f\|_p \le C_2(p) \sum_{n=1}^\infty n^{-1/2} a_n(f), \quad 2 \le p < \infty;$$

$$C_3(p) \sup_n n^{1/2} a_n(f) \le \|f\|_p \le C_4(p) \sum_{n=1}^\infty n^{1/p-1} a_n(f), \quad 1 < p \le 2.$$

The following theorem is a corollary of the above Theorem 2.3.

Theorem 2.4. *Let Ψ be a quasi-greedy basis of the L_p space, $1 < p < 2$, $2 < p < \infty$. Then for any set of indices Λ*

$$\|S_\Lambda(f)\|_p \le C(p)|\Lambda|^{h(p)}\|f\|_p, \quad h(p) := |1/p - 1/2|.$$

Proof. Let $m := |\Lambda|$. Using Theorem 2.3 we get for $1 < p < 2$

$$\|S_\Lambda(f)\|_p \le C_4(p) \sum_{n=1}^m n^{1/p-1} a_n(S_\Lambda(f))$$

$$= C_4(p) \sum_{n=1}^m n^{1/p-3/2}(n^{1/2} a_n(S_\Lambda(f))) \le C_4(p) \sum_{n=1}^m n^{1/p-3/2}(n^{1/2} a_n(f))$$
$$\le C_5(p) m^{1/p-1/2} \sup_n n^{1/2} a_n(f) \le C_5(p) C_3(p)^{-1} m^{1/p-1/2} \|f\|_p.$$

Again using Theorem 2.3 we obtain for $2 < p < \infty$

$$\|S_\Lambda(f)\|_p \le C_2(p) \sum_{n=1}^m n^{-1/2} a_n(S_\Lambda(f))$$

$$= C_2(p) \sum_{n=1}^m n^{-1/2-1/p}(n^{1/p} a_n(S_\Lambda(f))) \le C_2(p) \sum_{n=1}^m n^{-1/2-1/p}(n^{1/p} a_n(f))$$
$$\le C_6(p) m^{1/2-1/p} \sup_n n^{1/p} a_n(f) \le C_6(p) C_1(p)^{-1} m^{1/2-1/p} \|f\|_p.$$

□

It is pointed out in [6] that Theorem 2.3 implies the following inequality for a quasi-greedy basis Ψ of L_p

$$v(m,\Psi) \le C(p)m^{h(p)}, \quad 1<p<\infty. \tag{2.11}$$

Using inequality (2.11) in Theorem 2.2 we obtain the following Theorem 2.5.

Theorem 2.5. *Let Ψ be a quasi-greedy basis of L_p, $1<p<\infty$. Then for each $f\in L_p$*

$$\|f-G_m(f)\|_p \le C(\Psi,p)m^{h(p)}\tilde{\sigma}_m(f), \quad h(p):=|1/2-1/p|.$$

We now give a proof of Theorem 1.1 from the Introduction.

Proof. The first part of the proof goes along the lines of proof of Theorem 2.1. We use the notation from that proof. By Theorem 2.4 we obtain

$$w(m) \le C(p)m^{h(p)}. \tag{2.12}$$

Thus (2.4) gives

$$\|f-S_P(f)\|_p \le (1+C(p)m^{h(p)})\|f-p_m(f)\|_p. \tag{2.13}$$

Next, using Theorem 2.3 and our assumption that Ψ is a quasi-greedy basis of L_p we obtain for $1<p<2$

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f-p_m(f))\|_p \le C_4(p)\sum_{n=1}^{m} n^{1/p-1}a_n(S_{Q\setminus P}(f-p_m(f)))$$

$$\le C_4(p)\sum_{n=1}^{m} n^{1/p-1}a_n(G_{|Q\setminus P|}(f-p_m(f)))$$

$$= C_4(p)\sum_{n=1}^{m} n^{1/p-3/2}(n^{1/2}a_n(G_{|Q\setminus P|}(f-p_m(f))))$$

$$\le C_7(p)m^{1/p-1/2}\sup_n n^{1/2}a_n(G_{|Q\setminus P|}(f-p_m(f)))$$

$$\le C_7(p)C_3(p)^{-1}m^{1/p-1/2}\|G_{|Q\setminus P|}(f-p_m(f))\|_p$$

$$\le C_8(p)Km^{1/p-1/2}\|f-p_m(f)\|_p. \tag{2.14}$$

In the same way we treat the case $2 < p < \infty$.

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f - p_m(f))\|_p \leq C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f - p_m(f)))$$

$$\leq C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(G_{|Q\setminus P|}(f - p_m(f)))$$

$$= C_2(p) \sum_{n=1}^{m} n^{-1/2-1/p} (n^{1/p} a_n(G_{|Q\setminus P|}(f - p_m(f))))$$

$$\leq C_9(p) m^{1/2-1/p} \sup_n n^{1/p} a_n(G_{|Q\setminus P|}(f - p_m(f)))$$

$$\leq C_9(p) C_1(p)^{-1} m^{1/2-1/p} \|G_{|Q\setminus P|}(f - p_m(f))\|_p$$

$$\leq C_{10}(p) K m^{1/p-1/2} \|f - p_m(f)\|_p. \tag{2.15}$$

For the $S_{P\setminus Q}(f)$ we have for $1 < p < 2$

$$\|S_{P\setminus Q}(f)\|_p \leq C_4(p) \sum_{n=1}^{m} n^{1/p-1} a_n(S_{P\setminus Q}(f))$$

$$\leq C_4(p) \sum_{n=1}^{m} n^{1/p-1} a_n(S_{Q\setminus P}(f)) = C_4(p) \sum_{n=1}^{m} n^{1/p-1} a_n(S_{Q\setminus P}(f - p_m(f)))$$

which has been estimated in (2.14)

$$\leq C_8(p) K m^{1/p-1/2} \|f - p_m(f)\|_p. \tag{2.16}$$

In the same way we obtain the bound in the case $2 < p < \infty$

$$\|S_{P\setminus Q}(f)\|_p \leq C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{P\setminus Q}(f))$$

$$\leq C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f)) = C_2(p) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f - p_m(f)))$$

which has been estimated in (2.15)

$$\leq C_{10}(p) K m^{1/2-1/p} \|f - p_m(f)\|_p. \tag{2.17}$$

Combining (2.13) – (2.17) we complete the proof of Theorem 1.1.

$\square$

3 Lebesgue-type inequalities for Weak Greedy Algorithm

The greedy approximant $G_m(f)$ considered above was defined as the sum

$$\sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}$$

of the expansion terms with the m biggest (in absolute value) coefficients. In this section we make a remark on a more flexible way of construction of a greedy approximant. The rule of choosing the expansion terms for approximation will be weaker than in the greedy algorithm $G_m(\cdot)$. Instead of taking m terms with the biggest coefficients we now take m terms with near biggest coefficients. We proceed to a formal definition of the Weak Greedy Algorithm with regard to a basis Ψ (see [5], p. 271).

Let $t \in (0,1]$ be a fixed parameter. For a given basis Ψ and a given $f \in X$ denote $\Lambda_m(t)$ any set of m indices such that

$$\min_{k\in\Lambda_m(t)} |c_k(f)| \geq t \max_{k\notin\Lambda_m(t)} |c_k(f)| \tag{3.1}$$

and define

$$G_m^t(f) := G_m^t(f,\Psi) := \sum_{k\in\Lambda_m(t)} c_k(f)\psi_k.$$

We call it the Weak Greedy Algorithm with the weakness parameter t.

Results similar to those proved in Section 2 for the greedy approximant $G_m(f)$ can be proved for the weak greedy approximant $G_m^t(f)$. A generalization of Theorem 2.1 is straightforward.

Theorem 3.1. *Let Ψ be a quasi-greedy basis of X satisfying assumption (2.1) with the property: For any set of indices Λ*

$$\|S_\Lambda(f)\| \leq w(|\Lambda|)\|f\|.$$

Then for each $f \in X$

$$\|f - G_m^t(f)\| \leq C(K,t)v(m)w(m)\sigma_m(f).$$

It is well known that in the case of unconditional basis Ψ the function $w(m)$ is uniformly bounded: $w(m) \leq C(\Psi)$. In this case Theorem 3.1 implies the following Corollary 3.1.

Corollary 3.1. *Let Ψ be an unconditional basis of X satisfying assumption (2.1). Then for each $f \in X$*

$$\|f - G_m^t(f)\| \leq C(\Psi, t)v(m)\sigma_m(f).$$

Using inequality (2.11) we obtain the following version of Corollary 3.1 for the L_p spaces.

Corollary 3.2. *Let $1 < p < \infty$ and let Ψ be an unconditional basis of X. Then for each $f \in X$*

$$\|f - G_m^t(f)\|_p \leq C(\Psi, t, p)m^{|1/2-1/p|}\sigma_m(f)_p.$$

The following analog of Theorem 2.2 holds for the weak greedy approximant $G_m^t(f)$.

Theorem 3.2. *Let Ψ be a quasi-greedy basis of X satisfying assumption (2.1). Then for each $f \in X$*

$$\|f - G_m^t(f)\| \leq C(K, t)v(m)\tilde{\sigma}_m(f).$$

Proof. The proof of this theorem repeats the proof of Theorem 2.2. It defers only at one step. Instead of the inequality in (2.10)

$$\|G_{|Q\setminus B|}(f - S_B(f))\| \leq K\|f - S_B(f)\|$$

that is a direct corollary of the definition of a quasi-greedy basis we use the following known result (see [5], p. 272).

Theorem 3.3. *Let Ψ be a quasi-greedy basis. Then for a fixed $t \in (0, 1]$ and any m we have for any $f \in X$*

$$\|G_m^t(f)\| \leq C(\Psi, t)\|f\|.$$

$\square$

Using Theorem 3.3 we can prove the following analogs of Theorems 2.5 and 1.1.

Theorem 3.4. *Let Ψ be a quasi-greedy basis of L_p, $1 < p < \infty$. Then for each $f \in L_p$*

$$\|f - G_m^t(f)\|_p \leq C(\Psi, p, t)m^{|1/2-1/p|}\tilde{\sigma}_m(f).$$

Theorem 3.5. *Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have*

$$\|f - G_m^t(f)\|_p \leq C(\Psi, p, t) m^{|1/2-1/p|} \sigma_m(f)_p. \tag{3.2}$$

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