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ON THE CONSTRUCTION OF BASES AND FRAMES FOR SPACES OF DISTRIBUTIONS

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Abstract. We introduce a new method for constructing bases and frames for general distribution spaces and employ it to the construction of bases for modulation spaces on $\mathbb{R}^n$ and frames for Triebel-Lizorkin and Besov spaces on the sphere. Conceptually, our scheme allows the freedom to prescribe the nature, form, or some properties of the constructed basis or frame elements. For instance, they can be linear combinations of a small fixed number of shifts and dilates of any sufficiently smooth and rapidly decaying function. On the sphere, our frame elements consist of smooth functions supported on small shrinking caps.

1. Introduction

Bases and frames are a workhorse in Harmonic analysis in making various spaces of functions and distributions more accessible for study and utilization. Wavelets [18] are one of the most striking example of bases playing a pivotal role in Theoretical and Computational Harmonic analysis. The φ-transform of Frazier and Jawerth [5, 6, 7] is an example of a frame which has had a significant impact in Harmonic analysis. Orthogonal expansions were recently used for the development of frames of a similar nature in non-standard settings such as on the sphere [19, 20], interval [23, 15] and ball [24, 16] with weights, and in the context of Hermite [25] and Laguerre [12] expansions.

We begin with a general description of our "small perturbation argument" method for constructing bases and frames for spaces of distributions. Assume that H is a separable Hilbert space of functions (e.g. some L^2-space) and

$$\mathcal{S} \subset H \subset \mathcal{S}',$$

where $\mathcal{S}$ is a linear space of test functions and $\mathcal{S}'$ is the associated space of distributions. Suppose

$$L \subset \mathcal{S}'$$

is a quasi-Banach space of distributions with associated sequence space $\ell(\mathcal{X})$ which is a quasi-Banach space as well. Targeted spaces L are the modulation spaces on $\mathbb{R}^n$, the homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces on $\mathbb{R}^n$, and Triebel-Lizorkin and Besov spaces on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, unit ball in $\mathbb{R}^n$, and interval with weights as well as Triebel-Lizorkin and Besov spaces in the context of Hermite and Laguerre expansions. We consider two scenarios:

(i) there is an orthonormal basis $\Psi = \{\psi_\xi\}_{\xi\in\mathcal{X}}$ in H which provides an isomorphism between L and $\ell(\mathcal{X})$, or

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(ii) there is a frame $\Psi = \{\psi_\xi\}_{\xi\in\mathcal{X}}$ in H which allows to characterize L in terms of $\ell(\mathcal{X})$.

The central idea of our method is to construct a new system $\Theta = \{\theta_\xi\}_{\xi\in\mathcal{X}} \subset H$ which approximates Ψ sufficiently well in a specific sense, while at the same time the nature, form or some specific properties of the elements $\{\theta_\xi\}$ can be prescribed in advance. To make this scheme work we rely on two basic principles: Localization and Approximation. The *measure of localization* is in terms of the size of the various inner products of the form $\langle\psi_\xi, \psi_\eta\rangle$, $\langle\theta_\eta, \psi_\xi\rangle$, $\langle\psi_\xi, \theta_\eta\rangle$, more precisely, in terms of boundedness of the respective operators on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$. The *measure of approximation* is in terms of the size of the inner products of the form $\langle\psi_\eta, \psi_\xi - \theta_\xi\rangle$, $\langle\psi_\eta - \theta_\eta, \psi_\xi\rangle$. In fact, the critical step is to construct $\{\theta_\xi\}$ so that the operators with matrices

$$(\langle\psi_\eta, \psi_\xi - \theta_\xi\rangle)_{\xi,\eta\in\mathcal{X}} \quad \text{and} \quad (\langle\psi_\eta - \theta_\eta, \psi_\xi\rangle)_{\xi,\eta\in\mathcal{X}}$$

have sufficiently small norms on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$. The good localization and approximation properties of the new system Θ will guarantee that it is a basis or frame for the distribution spaces of interest.

The goal of this paper is two-fold: First, to develop our "small perturbation argument" method for construction of bases and frames in a general setup of distribution spaces, and second, to apply these results for developing new bases and frames for specific spaces of distributions. Choosing from various possible applications, we consider two key examples that best demonstrate the versatility of our general scheme. Building upon the Wilson basis we shall construct new bases for the modulation spaces on $\mathbb{R}^n$. The emphasis, however, will be on the construction of frames for Triebel-Lizorkin and Besov spaces on the sphere with elements supported on small shrinking caps. These frames are reminiscent of compactly supported wavelets on $\mathbb{R}^n$. The situation in our second example is much more complicated than on $\mathbb{R}^n$ since there are no dilation or translation operators on the sphere. Other meaningful applications of our scheme would be to the construction of frames on the interval and ball with weights, and in the context of Hermite and Laguerre expansions, which we shall not pursue here.

A relevant theme is the study of the localization and self-localization of frames, initiated by Gröchenig in [9, 10]. Our understanding of localization is different but related to the one in [9, 10]. Our idea of using the basic principles of localization and approximation mentioned above for constructing bases and frames for spaces of distributions has its roots in our previous developments, where bases and frames were constructed for Triebel-Lizorkin and Besov spaces on $\mathbb{R}^n$. Most of our previous results on bases and frames from [13, 14, 22] can now be derived as applications of the general theory developed in this article.

The paper is organized as follows: In §2 we develop our general method for construction of bases (§2.2) and frames (§2.3) for distribution spaces. In §3 we apply the general scheme from §2.2 to the construction of a basis for modulation spaces. In §4 we make an application of our general results from §2.3 to the construction of frames for the Triebel-Lizorkin and Besov spaces on the sphere.

Some useful notation: We shall denote $|x| := (\sum_i |x_i|^2)^{1/2}$ for $x \in \mathbb{R}^n$. Positive constants will be denoted by c, c_1, c_2, $\dots$ and they will be allowed to vary at every occurrence; $a \sim b$ will stand for $c_1 a \le b \le c_2 a$.

2. General scheme for construction of bases and frames

2.1. **The setting.** We assume that H is a separable complex Hilbert space (of functions) and $\mathcal{S} \subset H$ is a linear subspace (of test functions) furnished with a locally convex topology induced by a sequence of norms or semi-norms. Let $\mathcal{S}'$ be the dual of $\mathcal{S}$ consisting of all continuous linear functionals on $\mathcal{S}$. We also assume that $H \subset \mathcal{S}'$. The pairing of $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ will be denoted by $\langle f, \phi\rangle := f(\overline{\phi})$ and we assume that it is consistent with the inner product $\langle f, g\rangle$ in H. Typical examples are

(a) $H := L^2(\mathbb{R}^n)$, $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ is the Schwartz class on $\mathbb{R}^n$, and $\mathcal{S}'$ is the dual space of all tempered distributions on $\mathbb{R}^n$;

(b) $H := L^2(\mathbb{R}^n)$, $\mathcal{S} = \mathcal{S}_\infty(\mathbb{R}^n)$ is the set of all functions ϕ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\int \phi(x)x^\alpha = 0$ for $\alpha \in \mathbb{Z}^n_+$, and $\mathcal{S}'$ is its dual;

(c) $H := L^2(\mathbb{S}^n)$, $\mathcal{S} := C^\infty(\mathbb{S}^n)$ with $\mathbb{S}^n$ being the unit sphere in $\mathbb{R}^{n+1}$, and $\mathcal{S}'$ is its dual;

(d) $H := L^2(B^n, \mu)$, where B is the unit ball in $\mathbb{R}^n$ and $d\mu := (1 - |x|)^{\gamma-1/2}dx$, $\mathcal{S} := C^\infty(B^n)$, and $\mathcal{S}'$ is its dual;

(e) $H := L^2(I, \mu)$, where $I := I_1 \times \cdots \times I_n$ is a box in $\mathbb{R}^n$ and μ is a product Jacobi measure on I, $\mathcal{S} := C^\infty(I)$, and $\mathcal{S}'$ is its dual.

Our next assumption is that $L \subset \mathcal{S}'$ with norm $\|\cdot\|_L$ is a quasi-Banach space of distributions, which is continuously embedded in $\mathcal{S}'$. Further, we assume that $\mathcal{S} \subset H \cap L$ and $\mathcal{S}$ is dense in H and L with respect to their respective norms.

We also assume that $\ell(\mathcal{X})$ with norm $\|\cdot\|_{\ell(\mathcal{X})}$ is an associated to L quasi-Banach space of complex-valued sequences with domain a countable index set $\mathcal{X}$. Coupled with a basis or frame Ψ the sequence space $\ell(\mathcal{X})$ will be utilized for characterization of the space L. In addition to being a quasi-norm we assume that $\|\cdot\|_{\ell(\mathcal{X})}$ obeys the conditions:

(i) For any $\xi \in \mathcal{X}$ the projections $P_\xi : \ell(\mathcal{X}) \mapsto \mathbb{C}$ defined by $P_\xi(h) = h_\xi$ for $h = (h_\eta) \in \ell(\mathcal{X})$ are uniformly bounded on $\ell(\mathcal{X})$, i.e. $|h_\xi| \le c\|h\|_{\ell(\mathcal{X})}$ for $\xi \in \mathcal{X}$.

(ii) For any sequence $(h_\xi)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ one has $\|(h_\xi)\|_{\ell(\mathcal{X})} = \|(|h_\xi|)\|_{\ell(\mathcal{X})}$.

(iii) If the sequences $(h_\xi)_{\xi\in\mathcal{X}}, (g_\xi)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ and $|h_\xi| \le |g_\xi|$ for $\xi \in \mathcal{X}$, then $\|(h_\xi)\|_{\ell(\mathcal{X})} \le c\|(g_\xi)\|_{\ell(\mathcal{X})}$.

(iv) Compactly supported sequences are dense in $\ell(\mathcal{X})$.

2.2. Construction of bases for spaces of distributions.

2.2.1. *The old basis.* Given spaces $\mathcal{S} \subset H \subset \mathcal{S}'$, L, and $\ell(\mathcal{X})$ as described in §2.1 with $\mathcal{X}$ a countable index set, we assume that $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ is an orthonormal basis for H, that is, $\langle \psi_\xi, \psi_\eta\rangle = \delta_{\xi,\eta}$ for $\xi, \eta \in \mathcal{X}$, and any $f \in H$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } H \quad \text{and} \quad \|f\|_H = \|(\langle f, \psi_\xi\rangle)_\xi\|_{\ell^2(\mathcal{X})}. \tag{2.1}$$

We also assume that Ψ is a basis for the space L in the following sense:

(a) Every $f \in L$ has a unique representation in terms of $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ and

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{in } L. \tag{2.2}$$

(b) The operator $S_\Psi f := (\langle f, \psi_\xi\rangle)_{\xi\in\mathcal{X}}$ is bounded as an operator from L to $\ell(\mathcal{X})$.

(c) For any sequence $h \in \ell(\mathcal{X})$ the operator $T_\Psi h := \sum_{\xi\in\mathcal{X}} h_\xi \psi_\xi$ is well defined and bounded as an operator from $\ell(\mathcal{X})$ to L.

Consequently, for any $f \in L$

$$c_1\|f\|_L \le \|(\langle f, \psi_\xi\rangle)\|_{\ell(\mathcal{X})} \le c_2\|f\|_L \tag{2.3}$$

for some constants $c_1, c_2 > 0$.

Remark 2.1. In (2.1)-(2.2) above and throughout the rest of this section when we write "in H" or "in L" it will always mean that the convergence of the respective series is *unconditional* in H or in L. For unconditional convergence and bases we refer the reader to [17].

2.2.2. *Construction of a new basis.* Our idea is to first construct by perturbing Ψ a new basis $\Theta = \{\theta_\xi : \xi \in \mathcal{X}\}$ for H with elements $\theta_\xi \in H$ and then to show that under some additional localization and approximation conditions Θ is a basis for L. Since Ψ is a basis for H, we have

$$\theta_\xi = \sum_{\eta\in\mathcal{X}} \langle \theta_\xi, \psi_\eta\rangle \psi_\eta \quad \text{in } H. \tag{2.4}$$

Denote by $\mathbf{A}$ the transformation matrix

$$\mathbf{A} := (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad a_{\xi,\eta} := \langle \theta_\xi, \psi_\eta\rangle. \tag{2.5}$$

Our key assumption is that the operator $\mathbf{A}$ with matrix $\mathbf{A}$ is bounded and invertible on $\ell^2(\mathcal{X})$ and $\mathbf{A}^{-1}$ is also bounded on $\ell^2(\mathcal{X})$. Observe that if

$$\mathbf{D} = (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}} := (\langle \psi_\xi - \theta_\xi, \psi_\eta\rangle)_{\xi,\eta\in\mathcal{X}}, \tag{2.6}$$

then $\mathbf{A} = I - \mathbf{D}$ and, therefore, $\mathbf{A}^{-1}$ exists and is bounded on $\ell^2(\mathcal{X})$ if

$$\|\mathbf{D}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} < 1. \tag{2.7}$$

This is our *main assumption* in constructing Θ as a Riesz basis for H. The gist of our method is to approximate ψ_ξ by θ_ξ in such a way that $\mathbf{D}$ satisfies (2.7).

We shall show that under these conditions Θ is a Riesz basis for H. To proceed, let

$$\mathbf{A}^{-1} =: (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}} \tag{2.8}$$

and define

$$\tilde\theta_\xi := \sum_\eta \overline{b_{\eta,\xi}}\psi_\eta, \quad \xi \in \mathcal{X}. \tag{2.9}$$

Since $(\mathbf{A}^{-1})^* = \big(\overline{b_{\eta,\xi}}\big)_{\xi,\eta\in\mathcal{X}}$ is the adjoint matrix of $\mathbf{A}^{-1}$ and

$$\|(\mathbf{A}^{-1})^*\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} = \|\mathbf{A}^{-1}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} < \infty,$$

each vector row of $(\mathbf{A}^{-1})^*$ belongs to $\ell^2(\mathcal{X})$ and hence $\tilde\theta_\xi$ from (2.9) is well defined and $\tilde\theta_\xi \in H$. Evidently, $\overline{b_{\eta,\xi}} = \langle \tilde\theta_\xi, \psi_\eta\rangle$ and hence $\langle \psi_\eta, \tilde\theta_\xi\rangle = b_{\eta,\xi}$.

We set $\tilde\Theta := \{\tilde\theta_\xi : \xi \in \mathcal{X}\}$. Then it is easy to see that the pair $(\Theta, \tilde\Theta)$ is a biorthogonal system in H, i.e. $\langle \theta_\eta, \tilde\theta_\xi\rangle = \delta_{\xi,\eta}$. Indeed,

$$\langle \theta_\eta, \tilde\theta_\xi\rangle = \sum_\omega b_{\omega,\xi}\langle \theta_\eta, \psi_\omega\rangle = \sum_\omega a_{\eta,\omega}b_{\omega,\xi} = (\mathbf{A}\mathbf{A}^{-1})_{\eta,\xi} = \delta_{\eta,\xi}.$$

Proposition 2.2. *If $\mathbf{A}, \mathbf{A}^{-1}, \mathbf{A}^T, (\mathbf{A}^{-1})^T$ are bounded on $\ell^2(\mathcal{X})$, then Θ (with dual $\tilde\Theta$) is a Riesz basis for H. Consequently, for every $f \in H$*

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde\theta_\xi\rangle \theta_\xi \quad \text{in } H \quad \text{and} \tag{2.10}$$

$$c_1\|f\|_H \le \|(\langle f, \tilde\theta_\xi\rangle)_\xi\|_{\ell^2(\mathcal{X})} \le c_2\|f\|_H. \tag{2.11}$$

Proof. It is well know that (see e.g. [28]) a necessary and sufficient condition for $\Theta \subset H$ to be a Riesz basis for H is that Θ satisfies the conditions:

(i) Θ is complete in H (the closed span of Θ is all of H).

(ii) There exist constants $c', c'' > 0$ such that for any compactly supported sequence $h = (h_\xi)_{\xi\in\mathcal{X}}$ one has

$$c'\|h\|_{\ell^2(\mathcal{X})} \le \Big\| \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi \Big\|_H \le c''\|h\|_{\ell^2(\mathcal{X})}. \tag{2.12}$$

We shall first prove that for any $\xi \in \mathcal{X}$

$$\psi_\xi = \sum_{\omega\in\mathcal{X}} \langle \psi_\xi, \tilde\theta_\omega\rangle\theta_\omega = \sum_{\omega\in\mathcal{X}} b_{\xi,\omega}\theta_\omega \quad \text{in } H. \tag{2.13}$$

To this end we shall utilize this lemma:

Lemma 2.3. *The operator $Th := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ is well defined and bounded as an operator from $\ell^2(\mathcal{X})$ to H.*

Proof. Let $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers. Then by the boundedness of $\mathbf{A}^T$ on $\ell^2(\mathcal{X})$ and (2.1) we have

$$\begin{aligned} \|Th\|_H &= \Big\| \Big(\Big\langle \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi, \psi_\eta \Big\rangle\Big)_\eta \Big\|_{\ell^2(\mathcal{X})} = \Big\| \Big(\sum_{\xi\in\mathcal{X}} h_\xi\langle\theta_\xi, \psi_\eta\rangle\Big)_\eta \Big\|_{\ell^2(\mathcal{X})} \\ &\le c\|\mathbf{A}^T\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})}\|h\|_{\ell^2(\mathcal{X})} \le c\|h\|_{\ell^2(\mathcal{X})}. \end{aligned}$$

Since compactly supported sequences are dense in $\ell^2(\mathcal{X})$, it follows that the operator T is bounded as an operator from $\ell^2(\mathcal{X})$ to H. It also follows that for any sequence $\{h_\xi\} \in \ell^2(\mathcal{X})$ the series $\sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ converges unconditionally in $\ell^2(\mathcal{X})$. $\square$

We now prove (2.13). Since $\mathbf{A}^{-1} = (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}$ and $\|\mathbf{A}^{-1}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} < \infty$, we have $(b_{\xi,\omega})_\omega \in \ell^2(\mathcal{X})$ and applying Lemma 2.3 it follows that $g_\xi := \sum_{\omega\in\mathcal{X}} b_{\xi,\omega}\theta_\omega$ is a well defined element of H. On the other hand,

$$\langle g_\xi, \psi_\eta\rangle = \sum_{\omega\in\mathcal{X}} b_{\xi,\omega}\langle\theta_\omega, \psi_\eta\rangle = \sum_{\omega\in\mathcal{X}} b_{\xi,\omega}a_{\omega,\eta} = (\mathbf{A}^{-1}\mathbf{A})_{\xi,\eta} = \delta_{\xi,\eta},$$

yielding $g_\xi = \psi_\xi$. Hence, (2.13) holds.

As a basis $\Psi := \{\psi_\xi\}$ is complete in H and now (2.13) implies that $\Theta := \{\theta_\xi\}$ is complete in H as well.

We now turn to the proof of (2.12). Let $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers. Then by Lemma 2.3 $\big\| \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi \big\|_H \le c\|h\|_{\ell^2(\mathcal{X})}$, which gives the right-hand side estimate in (2.12).

For the other direction, denote briefly $f := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$. As was shown above the system $\Theta := \{\tilde\theta_\xi\}$, defined in (2.9), is the dual of Θ and hence for $\xi \in \mathcal{X}$

$$h_\xi = \langle f, \tilde\theta_\xi\rangle = \Big\langle f, \sum_{\eta\in\mathcal{X}} \overline{b_{\eta,\xi}}\psi_\eta \Big\rangle = \sum_{\eta\in\mathcal{X}} b_{\eta,\xi}\big\langle f, \psi_\eta\big\rangle,$$

which yields

$$\|h\|_{\ell^2(\mathcal{X})} \le \|(\mathbf{A}^{-1})^T\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})}\big\|\big(\langle f,\psi_\eta\rangle\big)\big\|_{\ell^2(\mathcal{X})} \le c\big\|\big(\langle f,\psi_\eta\rangle\big)\big\|_{\ell^2(\mathcal{X})} \le c\|f\|_H.$$

Here we used the boundedness of $(\mathbf{A}^{-1})^T$ on $\ell^2(\mathcal{X})$ and (2.1). Thus (2.12) is established and hence Θ is a Riesz basis. This in turn implies (2.10)-(2.11). $\square$

Our next aim is to show that under some reasonable conditions on $\mathbf{A}$ and $\mathbf{A}^{-1}$ the system Θ is an unconditional basis for L.

Theorem 2.4. *Let $\mathbf{A}$ and $\mathbf{A}^{-1}$ be bounded on $\ell^2(\mathcal{X})$ and assume that the operators $\mathbf{A}^T$ and $(\mathbf{A}^{-1})^T$ with matrices $\mathbf{A}^T$ and $(\mathbf{A}^{-1})^T$ are bounded on $\ell(\mathcal{X})$. Then Θ (with dual $\tilde{\Theta}$) is an unconditional basis for L in the following sense:*

(a) Every $f \in L$ has a unique representation in terms of $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ and

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\theta}_\xi\rangle\theta_\xi, \tag{2.14}$$

where by definition $\langle f,\tilde{\theta}_\xi\rangle := \sum_{\eta\in\mathcal{X}}\langle f,\psi_\eta\rangle\langle\psi_\eta,\tilde{\theta}_\xi\rangle$ and the series converges unconditionally in L.

(b) The operator $Sf := \{\langle f,\tilde{\theta}_\xi\rangle\}$ is bounded as an operator from L to $\ell(\mathcal{X})$.

(c) The operator $Th := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ is well defined and bounded as an operator from $\ell(\mathcal{X})$ to L.

Consequently, $T\circ S = I$ the identity on L and there exist constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_L \le \|(\langle f,\tilde{\theta}_\xi\rangle)_\xi\|_{\ell(\mathcal{X})} \le c_2\|f\|_L \quad \text{for} \quad f\in L. \tag{2.15}$$

Proof. We first prove the boundedness of the operator $S: L \to \ell(\mathcal{X})$. By (2.9) $\langle\psi_\xi,\tilde{\theta}_\eta\rangle = b_{\xi,\eta}$ and using that $(\mathbf{A}^{-1})^T$ is bounded on $\ell(\mathcal{X})$, we get

$$\begin{aligned}\|Sf\|_{\ell(\mathcal{X})} &= \|(\langle f,\tilde{\theta}_\xi\rangle)\|_{\ell(\mathcal{X})} \le \Big\|\Big(\sum_{\eta\in\mathcal{X}}\langle f,\psi_\eta\rangle\langle\psi_\eta,\tilde{\theta}_\xi\rangle\Big)_\xi\Big\|_{\ell(\mathcal{X})}\\ &\le \Big\|\Big(\sum_{\eta\in\mathcal{X}} b_{\eta,\xi}\langle f,\psi_\eta\rangle\Big)_\xi\Big\|_{\ell(\mathcal{X})} \le \|(\mathbf{A}^{-1})^T\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|(\langle f,\psi_\xi\rangle)\|_{\ell(\mathcal{X})}\\ &\le c\|(\langle f,\psi_\xi\rangle)\|_{\ell(\mathcal{X})} \le c\|f\|_L,\end{aligned} \tag{2.16}$$

where for the last inequality we used that the operator $S_\Psi : L \to \ell(\mathcal{X})$ is bounded according to our assumptions. Thus $S: L\to\ell(\mathcal{X})$ is bounded.

We now prove (c). Let first $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers. By (2.4) we have $\theta_\xi = \sum_\eta a_{\xi,\eta}\psi_\eta$ and hence

$$Th = \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi = \sum_{\eta\in\mathcal{X}}\Big(\sum_{\xi\in\mathcal{X}} a_{\xi,\eta}h_\xi\Big)\psi_\eta.$$

Then by condition (b) of Ψ (see (2.3)) and the boundedness of $\mathbf{A}^T$ on $\ell(\mathcal{X})$, we obtain

$$\|Th\|_L \le c\Big\|\Big(\sum_{\xi\in\mathcal{X}} a_{\xi,\eta}h_\xi\Big)_\eta\Big\|_{\ell(\mathcal{X})} \le c\|\mathbf{A}^T\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|h\|_{\ell(\mathcal{X})} \le c\|h\|_{\ell(\mathcal{X})}.$$

By condition (iv) on $\ell(\mathcal{X})$ compactly supported sequences are dense in $\ell(\mathcal{X})$, and hence the operator T can be extended uniquely as a bounded operator from $\ell(\mathcal{X})$ to L. More precisely, from above and conditions (ii)-(iv) on $\ell(\mathcal{X})$ it follows that for any sequence $h = (h_\xi)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ and any $\varepsilon > 0$ there exists a finite set of indices $\mathcal{F} \subset \mathcal{X}$ such that for every index set $\mathcal{F}' \subset \mathcal{X} \setminus \mathcal{F}$ we have

$$\Big\| \sum_{\xi\in\mathcal{F}'} h_\xi \theta_\xi \Big\|_L < \varepsilon.$$

This readily implies (see [17]) that the series $\sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ converges unconditionally in L. Thus T is a well defined and bounded operator from $\ell(\mathcal{X})$ to L.

It remains to show that (2.14) is valid. We define a new operator U on L by

$$Uf := \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\theta}_\xi\rangle \theta_\xi.$$

By (2.16) we have $\big(\langle f, \tilde{\theta}_\xi\rangle\big)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$ for $f \in L$. Then from the boundedness of the operator T it follows that the operator U is well defined and bounded on L.

On the other hand by (2.10), $Uf = f$ for $f \in H$ and hence for $f \in \mathcal{S}$, but $\mathcal{S}$ is dense in L. Therefore, $Uf = f$ for $f \in L$, i.e. (2.14) holds true.

The identity $T \circ S = I$ follows by (2.14), and (2.15) is an immediate consequence of (b) and (c). $\square$

2.3. **Construction of frames for spaces of distributions.**

2.3.1. *Frames in Hilbert spaces: Background.* Here we collect some basic facts from the theory of frames (cf. [2],[11]). Let H with inner product $\langle\cdot,\cdot\rangle$ be a separable Hilbert space. A family $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset H$, where $\mathcal{X}$ is a countable index set, is called a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|_H^2 \le \sum_{\xi\in\mathcal{X}} |\langle f, \psi_\xi\rangle|^2 \le B\|f\|_H^2 \quad \text{for } f \in H. \tag{2.17}$$

It is not hard to see that the *frame operator* $S : H \mapsto H$ defined by

$$Sf = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \tag{2.18}$$

is a bounded linear operator and $AI \le S \le BI$. Therefore, S is self-adjoint, S is invertible, and $B^{-1}I \le S^{-1} \le A^{-1}I$. Also,

$$S^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\psi_\xi\rangle S^{-1}\psi_\xi \quad \text{in } H. \tag{2.19}$$

The family $S^{-1}\Psi := \{S^{-1}\psi_\xi\}_{\xi\in\mathcal{X}}$ is a frame for H as well. Furthermore, for every $f \in H$

$$f = SS^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\psi_\xi\rangle \psi_\xi \quad \text{in } H \tag{2.20}$$

and

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle S^{-1}\psi_\xi \quad \text{in } H. \tag{2.21}$$

Thus Ψ and $S^{-1}\Psi$ provide (like Riesz bases) stable representations of all $f \in H$. However, unlike a basis, Ψ may be redundant and (2.20) is not necessarily a unique representation of f in terms of $\{\psi_\xi\}$. A similar observation holds for $S^{-1}\Psi$. The frame Ψ is termed a *tight frame* if $A = B$ in (2.17).

2.3.2. *The old frame.* We adhere to the setting describe in §2.1. In particular, we assume that $\mathcal{S} \subset H \subset \mathcal{S}'$, $L \subset \mathcal{S}'$, and $\ell(\mathcal{X})$ are spaces as described in §2.1 with $\mathcal{X}$ a countable index set. We also assume that for any $f \in H$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{in } H \quad \text{and} \quad \|f\|_H \sim \|(\langle f, \psi_\xi \rangle)\|_{\ell^2(\mathcal{X})}. \tag{2.22}$$

Therefore, $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ is a tight frame for H.

More importantly, we assume also that Ψ is a frame for L in the following sense:

A1. For any $f \in L$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{in } L. \tag{2.23}$$

A2. For any $f \in L$, $(\langle f, \psi_\xi \rangle)_\xi \in \ell(\mathcal{X})$, and

$$c_1 \|f\|_L \le \|(\langle f, \psi_\xi \rangle)\|_{\ell(\mathcal{X})} \le c_2 \|f\|_L. \tag{2.24}$$

Our aim is by using the idea of "small perturbation argument" to construct a new system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ with some prescribed features, which is a frame for L in the following sense:

Definition 2.5. *We say that $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset H$ is a frame for the space L with associated sequence space $\ell(\mathcal{X})$ if the following conditions are obeyed:*

B1. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_L \le \|(\langle f, \theta_\xi \rangle)\|_{\ell(\mathcal{X})} \le c_2 \|f\|_L \quad \text{for } f \in L, \tag{2.25}$$

where $\langle f, \theta_\xi \rangle$ is defined by $\langle f, \theta_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, \theta_\xi \rangle$.

B2. *The operator $S : L \mapsto L$ defined by*

$$Sf = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle \theta_\xi$$

is bounded and invertible on L; S^{-1} is also bounded on L and

$$S^{-1} f = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle S^{-1}\theta_\xi \quad \text{in } L.$$

B3. *There exist constants $c_3, c_4 > 0$ such that*

$$c_3 \|f\|_L \le \|(\langle f, S^{-1}\theta_\xi \rangle)\|_{\ell(\mathcal{X})} \le c_4 \|f\|_L \quad \text{for } f \in L, \tag{2.26}$$

where as above by definition $\langle f, S^{-1}\theta_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, S^{-1}\theta_\xi \rangle$.

B4. *For any $f \in L$*

$$f = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \theta_\xi = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle S^{-1}\theta_\xi \quad \text{in } L. \tag{2.27}$$

We recall our standing convention that "in H" or "in L" means that the convergence is unconditional in H or in L.

Observe that if L is a Hilbert space then properties **B2-4** are byproducts of **B1** (see §2.3.1). However, this is no longer true for more general spaces.

2.3.3. *Construction of a new frame.* The key of our method for constructing a new frame $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\}$ for L (as described above) is to build $\{\theta_\xi\}$ with appropriate localization and approximation properties with respect to the given tight frame Ψ. The localization of Θ will be measured in terms of the size of the inner products $\langle \psi_\xi, \psi_\eta \rangle$, $\langle \theta_\eta, \psi_\xi \rangle$, $\langle \psi_\xi, \theta_\eta \rangle$. More precisely, we construct $\{\theta_\xi\}$ so that the operators with matrices

$$\begin{aligned} \mathbf{A} &:= (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \qquad a_{\xi,\eta} := \langle \psi_\eta, \psi_\xi \rangle, \\ \mathbf{B} &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \qquad b_{\xi,\eta} := \langle \theta_\eta, \psi_\xi \rangle, \\ \mathbf{C} &:= (c_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \qquad c_{\xi,\eta} := \langle \psi_\eta, \theta_\xi \rangle, \end{aligned} \tag{2.28}$$

are bounded on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$. Notice that $\mathbf{C} = \mathbf{B}^*$ the adjoint of $\mathbf{B}$. The approximation property of Θ will be measured in terms of the size of the inner products $\langle \psi_\eta, \psi_\xi - \theta_\xi \rangle$, $\langle \psi_\eta - \theta_\eta, \psi_\xi \rangle$. Namely, we construct $\{\theta_\xi\}$ so that the operators with matrices

$$\begin{aligned} \mathbf{D} &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad d_{\xi,\eta} := \langle \psi_\eta, \psi_\xi - \theta_\xi \rangle, \\ \mathbf{E} &:= (e_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad e_{\xi,\eta} := \langle \psi_\eta - \theta_\eta, \psi_\xi \rangle, \end{aligned} \tag{2.29}$$

are bounded on $\ell^2(\mathcal{X})$ and $\ell(\mathcal{X})$ and, more importantly, for sufficiently small $\varepsilon > 0$

$$\|\mathbf{D}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \le \varepsilon, \quad \|\mathbf{E}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \le \varepsilon, \tag{2.30}$$

$$\|\mathbf{D}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \le \varepsilon, \quad \|\mathbf{E}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \le \varepsilon. \tag{2.31}$$

Notice that $\mathbf{E} = \mathbf{D}^*$.

Before we treat the case of general distribution spaces, we shall give sufficient conditions which guarantee that the new system Θ is a frame for the Hilbert space H itself.

Proposition 2.6. *As above, let $\Psi = \{\psi_\xi\}_{\xi\in\mathcal{X}}$ be a frame for the Hilbert space H such that* (2.22) *holds. Suppose $\Theta = \{\theta_\xi\}_{\xi\in\mathcal{X}} \subset H$ is constructed so that the operators with matrices $\mathbf{C}$ and $\mathbf{D}$ defined in* (2.28)-(2.29) *are bounded on $\ell^2(\mathcal{X})$ and for a sufficiently small $\varepsilon > 0$*

$$\|\mathbf{D}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \le \varepsilon. \tag{2.32}$$

Then Θ is a frame for H, that is, there exist constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_H \le \|(\langle f, \theta_\xi \rangle)_\xi\|_{\ell^2(\mathcal{X})} \le c_2\|f\|_H, \quad f \in H. \tag{2.33}$$

Proof. Note that $f = \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta \rangle \psi_\eta$ for $f \in H$ and hence

$$\begin{aligned} \|(\langle f, \theta_\xi \rangle)\|_{\ell^2(\mathcal{X})} &= \|(\sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, \theta_\xi \rangle)_\xi\|_{\ell^2(\mathcal{X})} \\ &\le \|\mathbf{C}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \|(\langle f, \psi_\xi \rangle)\|_{\ell^2(\mathcal{X})} \le c\|f\|_H. \end{aligned} \tag{2.34}$$

Thus the right-hand side estimate in (2.33) is established.

For the proof of the left-hand side of (2.33), we have using (2.17)

$$\begin{aligned} \|f\|_H &\le c\|(\langle f, \psi_\xi \rangle)\|_{\ell^2(\mathcal{X})} \\ &\le c\{\|(\langle f, \psi_\xi - \theta_\xi \rangle)\|_{\ell^2(\mathcal{X})} + \|(\langle f, \theta_\xi \rangle)\|_{\ell^2(\mathcal{X})}\}. \end{aligned} \tag{2.35}$$

Observe that

$$\begin{aligned} \|(\langle f, \psi_\xi - \theta_\xi \rangle)\|_{\ell^2(\mathcal{X})} &= \|(\sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, \psi_\xi - \theta_\xi \rangle)_\xi\|_{\ell^2(\mathcal{X})} \\ &\le \|\mathbf{D}\|_{\ell^2(\mathcal{X})\mapsto\ell^2(\mathcal{X})} \|(\langle f, \psi_\xi \rangle)\|_{\ell^2(\mathcal{X})} \le \varepsilon\|f\|_H. \end{aligned} \tag{2.36}$$

From (2.35)-(2.36) we obtain for sufficiently small $\varepsilon > 0$ ($\varepsilon < 1/c$ will do)

$$\|f\|_H \le \frac{c}{1-c\varepsilon}\|(\langle f,\theta_\xi\rangle)\|_{\ell^2(\mathcal{X})} \le c\|(\langle f,\theta_\xi\rangle)\|_{\ell^2(\mathcal{X})},$$

which confirms the left-hand side estimate in (2.33). □

We now come to the main result of this section.

Theorem 2.7. *Let $\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ be the old frame for H and L as described in §2.3.2. Suppose the system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset H$ is constructed so that the operators with matrices* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ *from* (2.28)-(2.29) *are bounded on $\ell(\mathcal{X})$ and* $\mathbf{C}$, $\mathbf{D}$ *are bounded on $\ell^2(\mathcal{X})$ as well. Then if for sufficiently small $\varepsilon > 0$ the matrices* $\mathbf{D}$, $\mathbf{E}$ *obey* (2.30)-(2.31)*, the sequence Θ is a frame for L in the sense of Definition 2.5.*

Most importantly, if $f \in \mathcal{S}'$, then $f \in L$ if and only if $(\langle f, S^{-1}\theta_\xi\rangle) \in \ell(\mathcal{X})$, and for $f \in L$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \textit{in } L \quad \textit{and} \quad \|f\|_L \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{\ell(\mathcal{X})}. \tag{2.37}$$

Proof. We first note that by Proposition 2.6 Θ is a frame for H.

We next prove that Θ obeys condition **B1**. From the definition of $\langle f, \theta_\xi\rangle$ (see Definition 2.5), the boundedness of $\mathbf{C}$, and (2.24) we infer

$$\begin{aligned} \|\big(\langle f,\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})} &= \Big\|\Big(\sum_{\eta\in\mathcal{X}} \langle f,\psi_\eta\rangle\langle\psi_\eta,\theta_\xi\rangle\Big)\Big\|_{\ell(\mathcal{X})} \\ &\le \|\mathbf{C}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|\big(\langle f,\psi_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c\|f\|_L, \end{aligned} \tag{2.38}$$

which confirms the right-hand side estimate in (2.25).

For the proof of the left-hand side of (2.25), we have by (2.24)

$$\|f\|_L \le c\|\big(\langle f,\psi_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c\|\big(\langle f,\psi_\xi-\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})} + c\|\big(\langle f,\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})}$$

and we next estimate the first term above using (2.31) and (2.24):

$$\begin{aligned} \|\big(\langle f,\psi_\xi-\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})} &= \Big\|\Big(\sum_{\eta\in\mathcal{X}} \langle f,\psi_\eta\rangle\langle\psi_\eta,\psi_\xi-\theta_\xi\rangle\Big)\Big\|_{\ell(\mathcal{X})} \\ &\le \|\mathbf{D}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|\big(\langle f,\psi_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c'\varepsilon\|f\|_L. \end{aligned}$$

Substituting this above, we get

$$\|f\|_L \le \frac{c}{1-cc'\varepsilon}\|\big(\langle f,\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})},$$

yielding the left-hand side estimate in (2.25) if $\varepsilon > 0$ is sufficiently small, namely, if $\varepsilon < 1/cc'$.

The following lemma will play a key role in the sequel.

Lemma 2.8. *The operators $Th := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ and $Vh := \sum_{\xi\in\mathcal{X}} h_\xi\psi_\xi$ are well defined and bounded as operators from $\ell(\mathcal{X})$ to L.*

Proof. We shall only prove the boundedness of T; the proof of the boundedness of V is easier and will be omitted. Let $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers. Then using (2.24) and the boundedness of $\mathbf{B}$, we get

$$\begin{aligned}\|Th\|_L &\le c\big\|\big(\langle \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi, \psi_\eta\rangle\big)_\eta\big\|_{\ell(\mathcal{X})} = c\big\|\big(\sum_{\xi\in\mathcal{X}} h_\xi\langle\theta_\xi,\psi_\eta\rangle\big)_\eta\big\|_{\ell(\mathcal{X})}\\ &\le c\|\mathbf{B}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|h\|_{\ell(\mathcal{X})} \le c\|h\|_{\ell(\mathcal{X})}.\end{aligned}$$

By condition (iv) on $\ell(\mathcal{X})$ compactly supported sequences are dense in $\ell(\mathcal{X})$ and, therefore, the operator T can be uniquely extended to a bounded operator from $\ell(\mathcal{X})$ to L. Furthermore, as in the proof of Theorem 2.4 it is easy to show that the series $\sum_{\xi\in\mathcal{X}} h_\xi\psi_\xi$ converges unconditionally in L. $\square$

We now prove that Θ satisfies **B2.** By definition $Sf = \sum_{\xi\in\mathcal{X}}\langle f,\theta_\xi\rangle\theta_\xi$, but by (2.38) we have $\big(\langle f,\theta_\xi\rangle\big)_{\xi\in\mathcal{X}} \in \ell(\mathcal{X})$. Therefore, by Lemma 2.8 the operator $S: L\to L$ is bounded.

The space L is a quasi-Banach space, but nevertheless it is easily seen that if $\|I-S\|_{L\mapsto L} < 1$, then S^{-1} exists and is bounded on L. In fact, S^{-1} can be constructed by the Neumann series, i.e. $S^{-1} = \sum_{k=0}^{\infty}(I-S)^k$. To prove that $\|I-S\|_{L\mapsto L} < 1$ for sufficiently small ε, let us denote $\mathbf{G} = (g_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}$, where $g_{\xi,\eta} := \langle (I-S)\psi_\eta,\psi_\xi\rangle$. Then, assuming that $\mathbf{G}$ is bounded on $\ell(\mathcal{X})$, we get

$$\begin{aligned}\|(I-S)f\|_L &\le c\|\big(\langle (I-S)f,\psi_\xi\rangle\big)\|_{\ell(\mathcal{X})} = c\Big\|\Big(\sum_{\eta\in\mathcal{X}}\langle f,\psi_\eta\rangle\langle (I-S)\psi_\eta,\psi_\xi\rangle\Big)\Big\|_{\ell(\mathcal{X})}\\ &\le c\|\mathbf{G}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|\big(\langle f,\psi_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c\|\mathbf{G}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|f\|_L. \qquad (2.39)\end{aligned}$$

Here for the equality we used that the operator $I-S$ is bounded on L. We next estimate $\|\mathbf{G}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}$. Evidently, we have

$$\langle S\psi_\eta,\psi_\xi\rangle = \sum_{\omega\in\mathcal{X}}\langle\psi_\eta,\theta_\omega\rangle\langle\theta_\omega,\psi_\xi\rangle \quad \text{and} \quad \langle\psi_\eta,\psi_\xi\rangle = \sum_{\omega\in\mathcal{X}}\langle\psi_\eta,\psi_\omega\rangle\langle\psi_\omega,\psi_\xi\rangle$$

and hence

$$\begin{aligned}g_{\xi,\eta} &= \langle\psi_\eta,\psi_\xi\rangle - \langle S\psi_\eta,\psi_\xi\rangle\\ &= \sum_{\omega\in\mathcal{X}}\langle\psi_\eta,\psi_\omega-\theta_\omega\rangle\langle\psi_\omega,\psi_\xi\rangle + \sum_{\omega\in\mathcal{X}}\langle\psi_\eta,\theta_\omega\rangle\langle\psi_\omega-\theta_\omega,\psi_\xi\rangle\\ &= (\mathbf{AD})_{\xi,\eta} + (\mathbf{EC})_{\xi,\eta}.\end{aligned}$$

Thus $\mathbf{G} = \mathbf{AD}+\mathbf{EC}$ and by the boundedness of the respective operators and (2.31)

$$\|\mathbf{G}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \le c(\|\mathbf{A}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|\mathbf{D}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} + \|\mathbf{E}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}\|\mathbf{C}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})}) \le c\varepsilon.$$

Substituting this in (2.39) we get $\|(I-S)f\|_L \le c''\varepsilon\|f\|_L$ and hence for sufficiently small ε we have $\|I-S\|_{L\mapsto L} \le c''\varepsilon < 1$ ($\varepsilon < 1/c''$ will do). Then the operator S^{-1} exists and is bounded on L.

For the rest of the proof of the theorem we need the following lemma:

Lemma 2.9. *The operators with matrices*

$$\begin{aligned}\mathbf{H} &:= \big(\langle\psi_\eta, S^{-1}\theta_\xi\rangle\big)_{\xi,\eta\in\mathcal{X}}, \quad &\mathbf{H}^* &:= \big(\langle S^{-1}\theta_\xi,\psi_\eta\rangle\big)_{\xi,\eta\in\mathcal{X}},\\ \mathbf{J} &:= \big(\langle\psi_\eta, S\psi_\xi\rangle\big)_{\xi,\eta\in\mathcal{X}}, \quad &\mathbf{J}_1 &:= \big(\langle\psi_\eta, S^{-1}\psi_\xi\rangle\big)_{\xi,\eta\in\mathcal{X}}\end{aligned}$$

are bounded on $\ell(\mathcal{X})$.

Proof. We shall only prove the boundedness of $\mathbf{H}$ and $\mathbf{H}^*$; the proof of the boundedness of $\mathbf{J}$ and $\mathbf{J}_1$ is simpler and will be omitted.

Let $d = (d_\xi)$ be a compactly supported sequence and set $f := \sum_{\xi\in\mathcal{X}} d_\xi \psi_\xi$. Then

$$\begin{aligned}(\mathbf{H}d)_\xi &= \sum_{\eta\in\mathcal{X}} d_\eta \langle \psi_\eta, S^{-1}\theta_\xi\rangle = \sum_{\eta\in\mathcal{X}} d_\eta \langle S^{-1}\psi_\eta, \theta_\xi\rangle = \langle \sum_{\eta\in\mathcal{X}} d_\eta S^{-1}\psi_\eta, \theta_\xi\rangle \\ &= \big\langle S^{-1}\big(\sum_{\eta\in\mathcal{X}} d_\eta\psi_\eta\big), \theta_\xi\big\rangle = \langle S^{-1}f, \theta_\xi\rangle = \sum_{\omega\in\mathcal{X}} \langle S^{-1}f, \psi_\omega\rangle\langle\psi_\omega, \theta_\xi\rangle.\end{aligned}$$

Here for the second equality we used that S^{-1} is self adjoint on H. Now, similarly as before we get

$$\|\mathbf{H}d\|_{\ell(\mathcal{X})} \le \|\mathbf{C}\|_{\ell(\mathcal{X})\mapsto\ell(\mathcal{X})} \|\big(\langle S^{-1}f, \psi_\omega\rangle\big)\|_{\ell(\mathcal{X})} \le c\|S^{-1}f\|_L \le c\|f\|_L \le c\|d\|_{\ell(\mathcal{X})}.$$

Here for the last inequality we used Lemma 2.8.

Since compactly supported sequences are dense in $\ell(\mathcal{X})$ then the operator $\mathbf{H}$ can be uniquely extended to a bounded operator on $\ell(\mathcal{X})$.

The proof of the boundedness of $\mathbf{H}^*$ goes along similar lines. Given a compactly supported sequence $d = (d_\xi)$, we set $g := \sum_{\eta\in\mathcal{X}} \overline{d_\eta}\psi_\eta$ and then

$$(\mathbf{H}^*d)_\xi = \sum_{\eta\in\mathcal{X}} d_\eta \langle S^{-1}\theta_\xi, \psi_\eta\rangle = \sum_{\eta\in\mathcal{X}} d_\eta \langle \theta_\xi, S^{-1}\psi_\eta\rangle = \big\langle \theta_\xi, S^{-1}\big(\sum_{\eta\in\mathcal{X}} \overline{d_\eta}\psi_\eta\big)\big\rangle.$$

As above, using the boundedness of S^{-1} on L and **B1**, we obtain

$$\|\mathbf{H}^*d\|_{\ell(\mathcal{X})} = \|\big(\langle S^{-1}g, \theta_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c\|S^{-1}g\|_L \le c\|g\|_L \le c\|\overline{d}\|_{\ell(\mathcal{X})} = c\|d\|_{\ell(\mathcal{X})}.$$

Here for the first and last equalities we used condition (ii) on $\ell(\mathcal{X})$. Now the boundedness of $\mathbf{H}^*$ follows as above. □

Just as in (2.38) the boundedness on $\ell(\mathcal{X})$ of the operator with matrix $\mathbf{H}$ from Lemma 2.9 implies

$$\|\big(\langle f, S^{-1}\theta_\xi\rangle\big)\|_{\ell(\mathcal{X})} \le c\|f\|_L \quad \text{for } f \in L.$$

Furthermore, the boundedness on $\ell(\mathcal{X})$ of the operator with matrix $\mathbf{H}^*$ defined in Lemma 2.9 yields that the operator

$$Uh := \sum_{\xi\in\mathcal{X}} h_\xi S^{-1}\theta_\xi$$

is bounded as an operator from $\ell(\mathcal{X})$ to L (see the proof of Lemma 2.8). Combining these two facts shows that the operator $S_\diamond^{-1}$ defined by

$$S_\diamond^{-1} f := \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle S^{-1}\theta_\xi \tag{2.40}$$

is well defined and bounded on L. On the other hand, by a well know property of frames (see (2.19)) for any $f \in H$

$$S^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle S^{-1}\theta_\xi. \tag{2.41}$$

Since by assumption $\mathcal{S} \subset H$ is dense in L, this leads to $S^{-1} = S_\diamond^{-1}$ on L. Therefore, representation (2.41) of S^{-1} holds on L as well. This completes the proof of **B2**.

We need one more lemma.

Lemma 2.10. *For any $f \in L$*

$$\langle Sf, \psi_\xi \rangle = \langle f, S\psi_\xi \rangle \quad \text{and} \quad \langle S^{-1}f, \psi_\xi \rangle = \langle f, S^{-1}\psi_\xi \rangle \quad \text{for} \quad \xi \in \mathcal{X}. \tag{2.42}$$

Proof. The proof relies on the fact that S and S^{-1} are self adjoint operators on H and $\mathcal{S} \subset H \cap L$ is dense in L.

We shall only prove the left-hand side identity in (2.42); the proof of the right-hand side identity is the same. Let $f \in L$ and choose a sequence $f_n \in \mathcal{S}$ so that $\|f - f_n\|_L \to 0$. Using that $\langle Sf_n, \psi_\xi \rangle = \langle f_n, S\psi_\xi \rangle$ as $f_n \in \mathcal{S}$, we get

$$|\langle Sf, \psi_\xi \rangle - \langle f, S\psi_\xi \rangle| \le |\langle S(f - f_n), \psi_\xi \rangle| + |\langle f - f_n, S\psi_\xi \rangle|. \tag{2.43}$$

By condition (i) on $\ell(\mathcal{X})$, (2.24), and the boundedness of S on L, it follows that

$$|\langle S(f-f_n), \psi_\xi \rangle| \le c\|(\langle S(f-f_n), \psi_\xi \rangle)\|_{\ell(\mathcal{X})} \le c\|S(f-f_n)\|_L \le c\|f-f_n\|_L. \tag{2.44}$$

By definition $\langle f - f_n, S\psi_\xi \rangle = \sum_{\eta \in \mathcal{X}} \langle f - f_n, \psi_\eta \rangle \langle \psi_\eta, S\psi_\xi \rangle$ and using again condition (i) on $\ell(\mathcal{X})$ and Lemma 2.9, we get

$$\begin{aligned} |\langle f - f_n, S\psi_\xi \rangle| &\le \Big\| \Big(\sum_{\eta \in \mathcal{X}} \langle f - f_n, \psi_\eta \rangle \langle \psi_\eta, S\psi_\xi \rangle \Big) \Big\|_{\ell(\mathcal{X})} \\ &\le \|\mathbf{J}\|_{\ell(\mathcal{X}) \mapsto \ell(\mathcal{X})} \big\| \big(\langle f - f_n, \psi_\eta \rangle \big) \big\|_{\ell(\mathcal{X})} \le c\|f - f_n\|_L \end{aligned}$$

We use this and (2.44) in (2.43) to obtain

$$|\langle Sf, \psi_\xi \rangle - \langle f, S\psi_\xi \rangle| \le c\|f - f_n\|_L \to 0,$$

which implies the left-hand side identity in (2.42). □

We are now prepared to prove that Θ obeys **B3-4.** Given $f \in L$, by definition $Sf = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle \theta_\xi$ and from $f = SS^{-1}f$ we arrive at

$$f = \sum_{\xi \in \mathcal{X}} \langle S^{-1}f, \theta_\xi \rangle \theta_\xi = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \theta_\xi \quad \text{in} \quad L, \tag{2.45}$$

where we used Lemma 2.10. Thus the left-hand side identity in (2.27) holds.

Similarly $f = S^{-1}Sf$ and using (2.41) in L and Lemma 2.10, we get

$$f = \sum_{\xi \in \mathcal{X}} \langle Sf, S^{-1}\theta_\xi \rangle S^{-1}\theta_\xi = \sum_{\xi \in \mathcal{X}} \langle f, SS^{-1}\theta_\xi \rangle S^{-1}\theta_\xi = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle S^{-1}\theta_\xi,$$

which gives the right-hand side identity in (2.27). Therefore, **B3** holds.

Going further, we have by definition $\langle f, S^{-1}\theta_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, S^{-1}\theta_\xi \rangle$ and using the boundedness of $\mathbf{H}$ (Lemma 2.9), we get

$$\|\big(\langle f, S^{-1}\theta_\xi \rangle\big)\|_{\ell(\mathcal{X})} \le \|\mathbf{H}\|_{\ell(\mathcal{X}) \mapsto \ell(\mathcal{X})} \|\big(\langle f, \psi_\eta \rangle\big)\|_{\ell(\mathcal{X})} \le c\|f\|_L,$$

which confirms the validity of the right-hand side estimate in (2.26).

In the other direction, by (2.45) $\langle f, \psi_\eta \rangle = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \langle \theta_\xi, \psi_\eta \rangle$ and hence

$$\begin{aligned} \|f\|_L \le c\|(\langle f, \psi_\eta \rangle)\|_{\ell(\mathcal{X})} &= c\Big\| \Big(\sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \langle \theta_\xi, \psi_\eta \rangle \Big)_\eta \Big\|_{\ell(\mathcal{X})} \\ &\le c\|\mathbf{B}\|_{\ell(\mathcal{X}) \mapsto \ell(\mathcal{X})} \big\| \big(\langle f, S^{-1}\theta_\xi \rangle \big) \big\|_{\ell(\mathcal{X})} \le c\big\| \big(\langle f, S^{-1}\theta_\xi \rangle \big) \big\|_{\ell(\mathcal{X})}. \end{aligned}$$

Thus **B3** is established.

Finally, observe that if $f \in \mathcal{S}'$ and $(\langle f, S^{-1}\theta_\xi \rangle) \in \ell(\mathcal{X})$, then by Lemma 2.8 $F := \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \theta_\xi \in L$. Since $g = \sum_{\xi \in \mathcal{X}} \langle g, S^{-1}\theta_\xi \rangle \theta_\xi$ for $g \in H$ and $\mathcal{S} \subset H \cap L$ is dense in L, then $F = f$. The proof of Theorem 2.7 is complete. □

3. New bases for modulation spaces

Here, we employ our scheme from §2.2 to the construction of bases for the modulation spaces $M_m^{p,q}$. We shall build upon the Wilson basis which is a basis for the modulation spaces.

3.1. **Modulation spaces: Background.** We first introduce some standard notation. The translation and modulation operators of a function f on $\mathbb{R}^n$ are defined by

$$T_y f(x) := f(x-y) \quad \text{and} \quad M_w f(x) := e^{2\pi i w\cdot x} f(x), \quad x, y, w \in \mathbb{R}^n. \tag{3.1}$$

By duality T_y and M_w extend to the space of tempered distribution $\mathcal{S}'$.

For $g \in L^2(\mathbb{R}^n)$, the short-time Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ with respect to g is defined by

$$V_g f(x,w) := \langle f, M_w T_x g\rangle := \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot w} dt. \tag{3.2}$$

The definition of $V_g f$ extends to distributions $f \in \mathcal{S}'$ as long as $g \in \mathcal{S}$ the class of rapidly decreasing test functions.

Definition 3.1. *Let $v_s(x,w) := (1+|x|+|w|)^s$, $s>0$, and suppose that $m \ge 0$ is a weight function on $\mathbb{R}^{2n}$ obeying*

$$m(x_1+x_2, w_1+w_2) \le c v_s(x_1,w_1) m(x_2,w_2). \tag{3.3}$$

Assume $1 \le p, q \le \infty$ and $g \in \mathcal{S}$, $g \not\equiv 0$. The modulation space $M_m^{p,q} = M_m^{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{M_m^{p,q}} := \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |V_g f(x,w)|^p m(x,w)^p\, dx\Big)^{q/p} dw\Big)^{1/q} < \infty \tag{3.4}$$

with the standard modification when $p=\infty$ or $q=\infty$.

Observe that $M_m^{p,q}$ is a Banach space and its definition is independent of the particular selection of the window function $g \in \mathcal{S}$. For the theory of modulation spaces, we refer the reader to [8].

3.2. **Wilson bases.** For any window function g on $\mathbb{R}$ and constants $\alpha, \beta > 0$, the countable collection

$$\mathcal{G}(g,\alpha,\beta) := \{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$$

is called a *Gabor system.*

Definition 3.2. *Given a window function ψ, assume $\mathcal{G}(\psi, \frac12, 1) \subset L^2(\mathbb{R})$. Then the Wilson system $\mathcal{W}(\psi)$ is defined by*

$$\mathcal{W}(\psi) := \{\psi_{k,j} : k \in \mathbb{Z}, j \in \mathbb{N}_0\},$$

where $\psi_{k,0} := T_k\psi$, $k \in \mathbb{Z}$, and

$$\psi_{k,j} := 2^{-1/2} T_{\frac{k}{2}}\big(M_j + (-1)^{k+j} M_{-j}\big)\psi, \quad (k,j) \in \mathbb{Z}\times\mathbb{N}.$$

Daubechies, Jaffard and Journé [1] have constructed an exponentially decaying window function ψ ($|\psi(x)| \le ce^{-\gamma|x|}$, $|\hat\psi(\xi)| \le ce^{-\gamma|\xi|}$) such that $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2(\mathbb{R})$. In what follows, we shall assume that $\mathcal{W}(\psi)$ is the exponentially localized Wilson basis from [1].

Using tensor products one easily constructs an orthonormal basis for $L^2(\mathbb{R}^n)$. Namely, for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $j = (j_1, \dots, j_n) \in \mathbb{N}_0^n$, set

$$\psi_{k,j}(x_1, \dots, x_n) := \prod_{i=1}^n \psi_{k_i,j_i}(x_i).$$

Then the multivariate Wilson system, defined by

$$\mathcal{W}(\psi) := \{\psi_{k,j} : k \in \mathbb{Z}^n, j \in \mathbb{N}_0^n\}, \tag{3.5}$$

is apparently an orthonormal basis for $L^2(\mathbb{R}^n)$.

For later use, we observe that for $k \in \mathbb{Z}^n, j \in \mathbb{N}_0^n$,

$$\begin{aligned}
\psi_{k,j}(x_1, \dots, x_n) &= a_j \prod_{i=1}^n T_{\frac{k_i}{2}} \big(M_{j_i} + (-1)^{k_i+j_i} M_{-j_i}\big)\psi(x_i) \\
&= a_j \sum_{\delta_j \in \{-1,1\}} \prod_{i=1}^n (-1)^{k_i+j_i} T_{\frac{k_i}{2}} M_{\delta_i j_i} \psi(x_i) \\
&= a_j \sum_{\delta \in \{-1,1\}^n} \delta^{k+j} T_{\frac{k}{2}} M_{\delta j} \psi^*(x),
\end{aligned} \tag{3.6}$$

where $\psi^*(x) := \psi(x_1) \cdots \psi(x_n)$, $\delta j := (\delta_1 j_1, \dots, \delta_n j_n)$, and $0 \le a_j \le 1$.

As is shown in [4], the Wilson basis is also an unconditional basis for the modulation spaces. Moreover, the membership of a tempered distribution f in $M_m^{p,q}$ can be characterized completely by the size of the coefficients $\langle f, \psi_{k,j}\rangle$, $(k,j) \in \mathbb{Z}^n \times \mathbb{N}_0^n$. The associated sequence space

$$\ell_m^{p,q} := \ell_m^{p,q}(\mathbb{Z}^n \times \mathbb{N}_0^n)$$

is defined as the set of all sequences $h = (h_{k,j})_{(k,j)\in\mathbb{Z}^n\times\mathbb{N}_0^n}$ such that

$$\|h\|_{\ell_m^{p,q}} := \Big(\sum_{j\in\mathbb{N}_0^n} \Big[\sum_{k\in\mathbb{Z}^n} |h_{k,j}|^p m(k/2,j)^p\Big]^{q/p}\Big)^{1/p} < \infty$$

with the usual modification when $p = \infty$ or $q = \infty$. Note that $\ell_m^{p,q}$ is a Banach space.

The characterization of modulation spaces via the Wilson basis reads as follows.

Theorem 3.3. [4] *Let $1 \le p, q \le \infty$ and suppose that $\mathcal{W}(\psi)$ is the Wilson basis, described above. Then there exist constants $c_1, c_2 > 0$ such that for every $f \in M_m^{p,q}$*

$$c_1 \|f\|_{M_m^{p,q}} \le \|(\langle f, \psi_{k,j}\rangle)_{k,j}\|_{\ell_m^{p,q}} \le c_2 \|f\|_{M_m^{p,q}}.$$

Moreover,

$$f = \sum_{(k,j)\in\mathbb{Z}^n\times\mathbb{N}_0^n} \langle f, \psi_{k,j}\rangle \psi_{k,j},$$

where the series converges unconditionally in the $M_m^{p,q}$-norm if $p, q < \infty$ and weak in $M_{1/v^s}^{\infty,\infty}$ otherwise.*

We refer the reader to [4], [8] for a detailed account of Wilson bases.

3.3. **Construction of new bases.** Let ψ^* (see (3.6)) be the exponentially localized function which generates the Wilson basis $\mathcal{W}(\psi) := \{\psi_{kj} : k \in \mathbb{Z}^n, j \in \mathbb{N}_0^d\}$. Given $s > 0$, fix an integer $M > 2n + s$. Evidently, there exists a constant $c^* > 0$ such that

$$|\partial^\alpha \psi^*(x)| \le \frac{c^*}{(1+|x|)^M}, \quad |\alpha| \le M, \tag{3.7}$$

where as usual $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_1})^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

For "small" $\varepsilon > 0$ (to be determined later on), we construct a function θ of some desired form or properties satisfying

$$|\partial^\alpha \psi^*(x) - \partial^\alpha \theta(x)| \le \frac{\varepsilon}{(1+|x|)^M}, \quad x \in \mathbb{R}^n, \quad |\alpha| \le M. \tag{3.8}$$

In analogy to (3.6), we set

$$\theta_{k,j}(x) := a_j \sum_{\delta \in \{-1,1\}^n} \delta^{k+j} M_{\delta j} T_{\frac{k}{2}} \theta(x) \tag{3.9}$$

and define the new system Θ by

$$\Theta := \{\theta_{k,j} : k \in \mathbb{Z}^n, j \in \mathbb{N}_0^n\}. \tag{3.10}$$

The main result in this section reads as follows.

Theorem 3.4. (*a*) *For a sufficiently small* ε *the system* Θ (*with dual* $\tilde\Theta := \{\tilde\theta_{k,j}\}$) *is a Riesz basis for* $L^2(\mathbb{R}^n)$.

(*b*) *For perhaps a different sufficiently small* $\varepsilon > 0$ *the system* Θ (*with dual* $\tilde\Theta$) *is a unconditional basis for* $M_m^{p,q}$, $1 \le p, q < \infty$. *In particular, for every* $f \in M_m^{p,q}$

$$f = \sum_{(k,j) \in \mathbb{Z}^n \times \mathbb{N}_0^n} \langle f, \tilde\theta_{k,j}\rangle \theta_{k,j},$$

where the convergence is unconditional in the $M_m^{p,q}$*-norm and*

$$c_1 \|f\|_{M_m^{p,q}} \le \|(\langle f, \tilde\theta_{k,j}\rangle)_{kj}\|_{\ell_m^{p,q}} \le c_2 \|f\|_{M_m^{p,q}}.$$

Before proving the theorem, we digress briefly and discuss the construction of functions θ which satisfy (3.8). Such constructions have been given in [13], where additionally θ has a number of vanishing moments which makes the construction more complex. To avoid unnecessary repetition we refer the reader to [13]. In particular, in [13] it is showed that if ϕ is sufficiently smooth and

$$|\partial^\alpha \phi(x)| \le c(1+|x|)^{-M'}, \quad |\alpha| \le N',$$

for sufficiently large M' and N', then using a finite linear combination of shifts and dilates of ϕ one can construct θ that satisfies (3.8). For instance, for any $\varepsilon > 0$ there can be constructed a function θ satisfying (3.8) which is a linear combination of finitely many shifts of $\phi(\cdot) := e^{-\gamma|\cdot|^2}$ or $\phi(\cdot) := (1 + \gamma|\cdot|^2)^{-N}$ with $N, \gamma > 0$ sufficiently large, or θ can be a compactly supported piecewise polynomial (spline) on $\mathbb{R}^n$. See also the relevant discussion in §4.4 below about the construction of compactly supported C^∞ function of the same type with vanishing moments.

Proof of Theorem 3.4. We shall utilize the scheme for constructing bases from §2.2 and, more precisely, Theorem 2.4 with $H := L^2(\mathbb{R}^n)$, $L := M_m^{p,q}$, and $\ell(\mathcal{X}) := \ell_m^{p,q}$ where $\mathcal{X} := \mathbb{Z}^n \times \mathbb{N}_0^n$. Denote

$$\mathbf{A} := \big(a_{(k,j),(l,m)}\big)_{(k,j),(l,m) \in \mathbb{Z}^n \times \mathbb{N}_0^n} \quad \text{with} \quad a_{(k,j),(l,m)} := \langle \theta_{k,j}, \psi_{l,m}\rangle, \quad \text{and} \tag{3.11}$$

$$\mathbf{D} := \big(d_{(k,j),(l,m)}\big)_{(k,j),(l,m)\in\mathbb{Z}^n\times\mathbb{N}_0^n}, \quad d_{(k,j),(l,m)} := \langle \psi_{k,j} - \theta_{k,j}, \psi_{l,m}\rangle. \tag{3.12}$$

Following Theorem 2.4 we need to show that the operators with matrices $\mathbf{A}$ and $\mathbf{A}^T$ are bounded on $\ell^2 := \ell^2(\mathbb{Z}^n \times \mathbb{N}_0^n)$ and on $\ell_m^{p,q}$, and also that

$$\|\mathbf{D}\|_{\ell^2\mapsto\ell^2} < 1, \qquad \|\mathbf{D}^T\|_{\ell^2\mapsto\ell^2} < 1, \tag{3.13}$$

$$\|\mathbf{D}\|_{\ell_m^{p,q}\mapsto\ell_m^{p,q}} < 1, \quad \|\mathbf{D}^T\|_{\ell_m^{p,q}\mapsto\ell_m^{p,q}} < 1. \tag{3.14}$$

To this end we need some preparation. We shall use the next lemma to estimate the inner products $\langle\theta_{k,j}, \psi_{l,m}\rangle$, $\langle\psi_{k,j} - \theta_{k,j}, \psi_{l,m}\rangle$.

Lemma 3.5. *Suppose that f, g are defined on $\mathbb{R}^n$ and for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le M$ we have*

$$|\partial^\alpha f(x)|, |\partial^\alpha g(x)| \le \frac{c^*}{(1+|x|)^M} \quad \text{and} \quad |\partial^\alpha f(x) - \partial^\alpha g(x)| \le \frac{\varepsilon}{(1+|x|)^M}. \tag{3.15}$$

Then for all $x, w \in \mathbb{R}^n$,

$$|V_g(f-g)(x,w)| \le \frac{c'\varepsilon}{(1+|x|+|w|)^M}, \tag{3.16}$$

where $c' > 0$ is independent of ε.

Proof. Conditions (3.15) with $\alpha = (0,\dots,0)$ readily imply

$$\begin{aligned} |V_g(f-g)(x,w)| &= \Big|\int_{\mathbb{R}^n} (f(t)-g(t))\overline{g(t-x)}e^{-2\pi i w\cdot t}\,dt\Big| \\ &\le c^*\varepsilon \int_{\mathbb{R}^n} \frac{dt}{(1+|t|)^M(1+|t-x|)^M} \le \frac{c\varepsilon}{(1+|x|)^M}. \end{aligned} \tag{3.17}$$

On the other hand, using Parseval's identity, we have

$$\begin{aligned} |V_g(f-g)(x,w)| &= \Big|\int_{\mathbb{R}^n} (f-g)^\wedge(\xi)\overline{\big(g(t-x)e^{2\pi i w\cdot t}\big)^\wedge(\xi)}\,d\xi\Big| \\ &= \Big|\int_{\mathbb{R}^n} (\hat f(\xi)-\hat g(\xi))e^{-2\pi i(\xi-w)\cdot x}\overline{\hat g(\xi-w)}\,d\xi\Big| \\ &\le \int_{\mathbb{R}^n} |\hat g(\xi)||\hat g(\xi+w) - \hat f(\xi+w)|\,d\xi. \end{aligned} \tag{3.18}$$

Here $\hat f(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}dx$. By the fact that $(1+|\xi|)^M \le c\sum_{|\alpha|\le M}|\xi^\alpha|$ and (3.15) we infer

$$\begin{aligned} (1+|\xi|)^M|\hat g(\xi) - \hat f(\xi)| &\le c\sum_{|\alpha|\le M} |\xi^\alpha(\hat g(\xi) - \hat f(\xi))| \\ &\le c\sum_{|\alpha|\le M} |\widehat{\partial^\alpha g}(\xi) - \widehat{\partial^\alpha f}(\xi)| \le c\sum_{|\alpha|\le M}\int_{\mathbb{R}^n} |\partial^\alpha g(t) - \partial^\alpha f(t)|\,dt \\ &\le c\varepsilon\int_{\mathbb{R}^n} \frac{1}{(1+|t|)^M}\,dt \le c\varepsilon. \end{aligned}$$

Therefore, $|\hat g(\xi) - \hat f(\xi)| \le \frac{c\varepsilon}{(1+|\xi|)^M}$. Exactly in the same way, $|\hat g(\xi)| \le \frac{c}{(1+|\xi|)^M}$. Combining these two estimate with (3.18), we obtain for $x, w \in \mathbb{R}^n$

$$|V_g(f-g)(x,w)| \le c\varepsilon\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^M(1+|\xi+w|)^M}\,d\xi \le \frac{c\varepsilon}{(1+|w|)^M}.$$

This estimate and (3.17) apparently yield (3.16). $\square$

We can now estimate the inner products under question.

Lemma 3.6. *For all* $(k,j),(l,m) \in \mathbb{Z}^n \times \mathbb{N}_0^n$ *we have*

$$|\langle \theta_{k,j}, \psi_{l,m}\rangle| \le \frac{c}{(1+|k-l|+|j-m|)^M} \tag{3.19}$$

and

$$|\langle \psi_{k,j} - \theta_{k,j}, \psi_{l,m}\rangle| \le \frac{c\varepsilon}{(1+|k-l|+|j-m|)^M}, \tag{3.20}$$

where $c>0$ *is independent of* ε.

Proof. We shall only prove (3.20); the proof of (3.19) is the same. It is easy to see that

$$\langle M_zT_yf, M_wT_xg\rangle = e^{-2\pi i w\cdot y}V_gf(x-y,w-z), \qquad x,y,w,z\in\mathbb{R}^n.$$

Then by (3.6) and (3.9) it follows that

$$\begin{aligned}|\langle \psi_{k,j} - \theta_{k,j}, \psi_{l,m}\rangle| &= \Big|a_ja_m \sum_{\delta_1,\delta_2\in\{-1,1\}^n} \delta_1^{k+j}\delta_2^{l+m}\big\langle M_{\delta_1 j}T_{\frac{k}{2}}(\psi^*-\theta), M_{\delta_2 m}T_{\frac{l}{2}}\psi^*\big\rangle\Big| \\ &\le \sum_{\delta_1,\delta_2\in\{-1,1\}^n} |V_{\psi^*}(\psi^*-\theta)(l/2-k/2, \delta_2 m - \delta_1 j)|\end{aligned}$$

and applying Lemma 3.5 we obtain

$$\begin{aligned}|\langle \psi_{k,j} - \theta_{k,j}, \psi_{l,m}\rangle| &\le \frac{c\varepsilon}{(1+|k-l|+|j-m|)^M} + \frac{c\varepsilon}{(1+|k-l|+|j+m|)^M} \\ &\le \frac{c\varepsilon}{(1+|k-l|+|j-m|)^M},\end{aligned}$$

where for the last inequality we used that $|j+m|\ge|j-m|$ if $j,m\in\mathbb{N}_0^n$. $\square$

We are now in a position to estimate the norms of $\mathbf{A}$, $\mathbf{A}^T$, $\mathbf{D}$, $\mathbf{D}^T$ on $\ell_m^{p,q}$ and ℓ^2. To this end we shall use the following inequality (see [8]): If $b=(b_{k,j})\in\ell_{v_s}^{1,1}$ and $h=(h_{k,j})\in\ell_m^{p,q}$, then

$$\|b*h\|_{\ell_m^{p,q}} \le c\|b\|_{\ell_{v_s}^{1,1}}\|h\|_{\ell_m^{p,q}}, \tag{3.21}$$

where the discrete convolution $b*h$ is defined as usual by

$$(b*h)_{k,j} = \sum_{l,m} b_{k-l,j-m}h_{l,m}.$$

To prove the boundedness of $\mathbf{A}$ on $\ell_m^{p,q}$, we observe that from (3.19) we have

$$|a_{(k,j),(l,m)}| \le cb_{k-l,j-m} \quad \text{for} \quad (k,j),(l,m)\in\mathbb{Z}^n\times\mathbb{N}_0^n,$$

where $b_{k,j} := \frac{1}{(1+|k|+|j|)^M}$. Hence, for any sequence $h=(h_{k,j})$

$$|(\mathbf{A}h)_{k,j}| = \Big|\sum_{l,m} a_{(k,j),(l,m)}h_{l,m}\Big| \le c\sum_{l,m} b_{k-l,j-m}|h_{l,m}| = c(b*|h|)_{k,j}.$$

But, it is easy to see that $\|(b_{k,j})\|_{\ell_{v_s}^{1,1}} \le c < \infty$ since $M>2n+s$. Consequently, by (3.21) we have for $h=(h_{k,j})\in\ell_m^{p,q}$

$$\|Ah\|_{\ell_m^{p,q}} \le c\|b\|_{\ell_{v_s}^{1,1}}\|h\|_{\ell_m^{p,q}} \le c\|h\|_{\ell_m^{p,q}}$$

and hence $\|\mathbf{A}\|_{\ell_m^{p,q}\mapsto\ell_m^{p,q}} \le c < \infty$. To proof the boundedness of $\mathbf{A}$ on ℓ^2 is easier. We omit the details.

Exactly as above, we get

$$\|\mathbf{D}\|_{\ell^{p,q}_m \mapsto \ell^{p,q}_m} \le c\varepsilon,$$

where the constant $c > 0$ is independent of ε. Hence, for sufficiently small $\varepsilon > 0$ we have $\|\mathbf{D}\|_{\ell^{p,q}_m \mapsto \ell^{p,q}_m} < 1$. By the same token, for sufficiently small $\varepsilon > 0$ $\|\mathbf{D}^T\|_{\ell^{p,q}_m \mapsto \ell^{p,q}_m} < 1$, $\|\mathbf{D}\|_{\ell^2 \mapsto \ell^2} < 1$, and $\|\mathbf{D}^T\|_{\ell^2 \mapsto \ell^2} < 1$. In turn, these inequalities lead to the boudedness of $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^T$ on $\ell^{p,q}_m$ and ℓ^2. Finally, we invoke Theorem 2.4 to complete the proof of Theorem 3.4. □

4. Frames with elements supported on shrinking caps on the sphere

In this section we utilize the scheme from §2.3 to the construction of frames for Triebel-Lizorkin (F) and Besov (B) spaces on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ $(n > 1)$ of the form $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, where $\mathcal{X} = \cup_{j=0}^\infty \mathcal{X}_j$ is a multilevel index set of points on $\mathbb{S}^n$ and for $\xi \in \mathcal{X}_j$ the frame element θ_ξ is supported on a spherical cap of radius $\sim 2^{-j}$ centered at ξ. The F- and B- spaces on the sphere are introduced and explored in [20] as a natural progression of the Littlewood-Paley theory on $\mathbb{S}^n$. These spaces are also characterized in [20] via frames with elements of nearly exponential localization, called "needlets". We next give a short account of the development in [20], which we shall build upon.

In contrast to §3 it will be convenient in this section to define the Fourier transform $\hat{f}$ of a function f on $\mathbb{R}$ by $\hat{f}(\xi) := \int_{\mathbb{R}} f(y)e^{-i\xi y}dy$.

4.1. **Spaces of distribution on the sphere: Background.**

Denote by $\mathcal{H}_\nu$ the space of all spherical harmonics of order ν on $\mathbb{S}^n$. As is well known the kernel of the orthogonal projector onto $\mathcal{H}_\nu$ is given by

$$\mathsf{P}_\nu(\xi\cdot\eta) = \frac{\nu+\lambda}{\lambda\omega_n} P_\nu^{(\lambda)}(\xi\cdot\eta), \quad \lambda = \lambda_n := \frac{n-1}{2}, \tag{4.1}$$

where ω_n is the hypersurface area of $\mathbb{S}^n$ and $P_\nu^{(\lambda)}$ is the Gegenbauer polynomial of degree ν normalized with $P_\nu^{(\lambda)}(1) = \binom{\nu+2\lambda-1}{\nu}$; $\xi\cdot\eta$ is the inner product of $\xi, \eta \in \mathbb{S}^n$.

Let $\mathcal{S} := C^\infty(\mathbb{S}^n)$ be the space of all test functions on $\mathbb{S}^n$ and $\mathcal{S}' := \mathcal{S}'(\mathbb{S}^n)$ be its dual, the space of all distributions on $\mathbb{S}^n$. The action of $f \in \mathcal{S}'$ on $\overline{\phi} \in \mathcal{S}$ is denoted by $\langle f, \phi\rangle := f(\overline{\phi})$.

For functions $\Phi \in L^\infty[-1,1]$ and $f \in L^1(\mathbb{S}^n)$ the *nonstandard convolution* $\Phi * f$ is defined by

$$\Phi * f(\xi) := \int_{\mathbb{S}^n} \Phi(\xi\cdot\sigma) f(\sigma)\, d\sigma,$$

where the integration is over $\mathbb{S}^n$, and it extends by duality from $\mathcal{S}$ to $\mathcal{S}'$.

To define the Triebel-Lizorkin and Besov spaces on the sphere, one first introduces a sequence of functions $\{\Phi_j\}$ of the form

$$\Phi_0 := \mathsf{P}_0 \quad \text{and} \quad \Phi_j := \sum_{\nu=0}^\infty \hat{a}\Big(\frac{\nu}{2^{j-1}}\Big)\mathsf{P}_\nu, \quad j \ge 1, \tag{4.2}$$

with $\hat{a}$ obeying the conditions:

$$\hat{a} \in C^\infty[0,\infty), \quad \operatorname{supp}\hat{a} \subset [1/2, 2], \tag{4.3}$$

$$|\hat{a}(t)| > c > 0 \quad \text{if } t \in [3/5, 5/3]. \tag{4.4}$$

Hence, Φ_j, $j = 0, 1, \dots$, are band limited.

Definition 4.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$. The Triebel-Lizorkin space $F_p^{sq} := F_p^{sq}(\mathbb{S}^n)$ is defined as the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{F_p^{sq}} := \Big\| \Big(\sum_{j=0}^{\infty} (2^{sj} |\Phi_j * f(\cdot)|)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{S}^n)} < \infty, \tag{4.5}$$

where the ℓ^q-norm is replaced by the sup-norm if $q = \infty$.

We note that as in the classical case on $\mathbb{R}^n$ by varying the indexes s, p, q one can recover most of the classical spaces on $\mathbb{S}^n$, e.g. $F_p^{02} = L^p(\mathbb{S}^n)$ if $1 < p < \infty$.

Definition 4.2. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. The Besov space $B_p^{sq} := B_p^{sq}(\mathbb{S}^n)$, is defined as the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{B_p^{sq}} := \Big(\sum_{j=0}^{\infty} \Big(2^{sj} \|\Phi_j * f\|_{L^p(\mathbb{S}^n)} \Big)^q \Big)^{1/q} < \infty \tag{4.6}$$

with the usual modification when $q = \infty$.

Remark. Observe that the above definitions of Triebel-Lizorkin and Besov spaces are independent of the specific selection of $\hat{a}$. For more details, see [20].

We refer the reader to [21] and [27] as general references for Triebel-Lizorkin and Besov spaces.

4.2. **Frame on $\mathbb{S}^n$ (Needlets).** In this part we slightly defer from [20]. Let $\hat{a}$ satisfy the conditions

$$\begin{aligned} &(i) \qquad \hat{a} \in C^\infty[0, \infty), \quad \hat{a} \geq 0, \quad \operatorname{supp} \hat{a} \subset [1/2, 2], \\ &(ii) \qquad \hat{a}(t) > c > 0, \quad \text{if } t \in [3/5, 5/3], \\ &(iii) \qquad \hat{a}^2(t) + \hat{a}^2(2t) = 1, \quad \text{if } t \in [1/2, 1] \end{aligned} \tag{4.7}$$

and hence,

$$\sum_{j=0}^{\infty} \hat{a}^2(2^{-j} t) = 1, \quad t \in [1, \infty). \tag{4.8}$$

We select $j_0 \geq -2$ so that $2^{j_0+1} \leq \lambda < 2^{j_0+2}$ ($\lambda := \frac{n-1}{2}$) and define the kernels $\{\Psi_j\}$ by $\Psi_{j_0} := \mathsf{P}_0$ and

$$\Psi_j := \sum_{\nu=0}^{\infty} \hat{a}\Big(\frac{\nu + \lambda}{2^j}\Big) \mathsf{P}_\nu, \quad j > j_0. \tag{4.9}$$

A Calderón type reproducing formula follows from (4.8)-(4.9): For any $f \in \mathcal{S}'$

$$f = \sum_{j=j_0}^{\infty} \Psi_j * \Psi_j * f \quad \text{in } \mathcal{S}'. \tag{4.10}$$

As in [20] (see also [19]) there exist a set $\mathcal{X}_j \subset \mathbb{S}^n$ ($j \geq j_0$) and weights $\{c_\xi\}_{\xi \in \mathcal{X}_j}$ such that the cubature formula

$$\int_{\mathbb{S}^n} f(\sigma) d\sigma \sim \sum_{\xi \in \mathcal{X}_j} c_\xi f(\xi) \tag{4.11}$$

is exact for all spherical polynomials of degree $\leq 2^{j+1}$. Here, in addition, $c_\xi \sim 2^{-jn}$ and the points in $\mathcal{X}_j$ are almost uniformly distributed, i.e. there exist constants

$c_2 > c_1 > 0$ such that $B_\xi(c_1 2^{-j}) \cap B_\eta(c_1 2^{-j}) = \emptyset$ whenever $\xi \neq \eta$, $\xi, \eta \in \mathcal{X}_j$, and $\mathbb{S}^n = \cup_{\xi \in \mathcal{X}_j} B_\xi(c_2 2^{-j})$, where $B_\xi(r) := \{\eta \in \mathbb{S}^n : d(\eta, \xi) < r\}$ with $d(\eta, \xi)$ being the geodesic distance between η, ξ on $\mathbb{S}^n$.

The jth level needlets are defined by

$$\psi_\xi(x) := c_\xi^{1/2} \Psi_j(\xi \cdot x), \quad \xi \in \mathcal{X}_j, \tag{4.12}$$

and the whole needlet system by

$$\Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}, \quad \text{where} \quad \mathcal{X} := \cup_{j=j_0}^{\infty} \mathcal{X}_j. \tag{4.13}$$

Here equal points from different levels $\mathcal{X}_j$ are regarded as distinct points of the index set $\mathcal{X}$.

By discretization of (4.10) using cubature formula (4.11) one arrives at the representation formula: For any $f \in \mathcal{S}'$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{in } \mathcal{S}'. \tag{4.14}$$

The same representation holds in L^p for functions $f \in L^p(\mathbb{S}^n)$ as well.

The key feature of the functions $\psi_\xi, \xi \in \mathcal{X}$, is their superb localization: For any $M > 0$ there exists a constant $c_M > 0$ such that

$$|\psi_\xi(x)| \leq c_M \frac{2^{jn/2}}{(1 + 2^j d(\xi, x))^M}, \quad x \in \mathbb{S}^n, \tag{4.15}$$

where as mentioned above $d(\xi, \eta) := \arccos(\xi \cdot \eta)$.

We next define the sequence spaces f_p^{sq} and b_p^{sq} associated to $\mathcal{X}$, where for $\xi \in \mathcal{X}_j$, G_ξ denotes the spherical cap $B_\xi(c_2 2^{-j})$, introduced above.

Definition 4.3. *Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$. Then $f_p^{sq} := f_p^{sq}(\mathcal{X})$ is defined as the space of all complex-valued sequences $h := (h_\xi)_{\xi \in \mathcal{X}}$ such that*

$$\|h\|_{f_p^{sq}} := \Big\| \Big(\sum_{\xi \in \mathcal{X}} \big[|G_\xi|^{-s/n - 1/2} |h_\xi| \mathbb{1}_{G_\xi}(\cdot) \big]^q \Big)^{1/q} \Big\|_{L^p} < \infty \tag{4.16}$$

with the usual modification for $q = \infty$. Here $|G_\xi|$ is the measure of G_ξ and $\mathbb{1}_{G_\xi}$ is the characteristic function of G_ξ.

Definition 4.4. *Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. Then $b_p^{sq} := b_p^{sq}(\mathcal{X})$ is defined as the space of all complex-valued sequences $h := (h_\xi)_{\xi \in \mathcal{X}}$ such that*

$$\|h\|_{b_p^{sq}} := \Big(\sum_{m=0}^{\infty} \Big[2^{j(s + n/2 - n/p)} \Big(\sum_{\xi \in \mathcal{X}_m} |h_\xi|^p \Big)^{1/p} \Big]^q \Big)^{1/q} < \infty \tag{4.17}$$

with the usual modification when $p = \infty$ or $q = \infty$.

Observe that $f_2^{02} = b_2^{02} = \ell^2(\mathcal{X})$ with equivalent norms.

The main result here asserts that Ψ is a frame for Triebel-Lizorkin and Besov spaces on the sphere in the sense of the following theorem.

Theorem 4.5. [20] *Let $s \in \mathbb{R}$ and $0 < p, q < \infty$.*

(a) If $f \in \mathcal{S}'$, then $f \in F_p^{sq}$ if and only if $(\langle f, \psi_\xi \rangle)_{\xi \in \mathcal{X}} \in f_p^{sq}$. Furthermore, for any $f \in F_p^{sq}$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \textit{and} \quad \|f\|_{F_p^{sq}} \sim \|(\langle f, \psi_\xi \rangle)\|_{f_p^{sq}}. \tag{4.18}$$

(*b*) *If* $f \in \mathcal{S}'$, *then* $f \in B_p^{sq}$ *if and only if* $(\langle f, \psi_\xi\rangle)_{\xi\in\mathcal{X}} \in b_p^{sq}$. *Furthermore, for any* $f \in B_p^{sq}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \psi_\xi \quad \text{and} \quad \|f\|_{B_p^{sq}} \sim \|(\langle f, \psi_\xi\rangle)\|_{B_p^{sq}}. \tag{4.19}$$

The convergence in (4.18) *and* (4.19) *is unconditional in* F_p^{sq} *and* B_p^{sq}, *respectively.*

Remark 4.6. A word of clarification is needed here. First, the result of Theorem 4.5 above is stated and proved in [20] for a pair of dual frames $\{\varphi_\xi\}$ and $\{\psi_\xi\}$. Here we need it in the case when $\varphi_\xi = \psi_\xi$. Second, in [20] it is only stated that the series in (4.18)-(4.19) converge in $\mathcal{S}'$, but it is allowed to have $p = \infty$ or $q = \infty$. It is easy to see that when $p, q < \infty$ the boundedness of the operator $T_\psi h := \sum_{\xi\in\mathcal{X}} h_\xi\psi_\xi$ as an operator from f_p^{sq} to F_p^{sq} or from b_p^{sq} to B_p^{sq}, proved in [20], implies that the series in (4.18) or (4.19) converge unconditionally in F_p^{sq} or B_p^{sq}, respectively. However, this is no longer true if $p = \infty$ or $q = \infty$ since $\mathcal{S}$ is not dense in F_p^{sq} and B_p^{sq} in this case.

4.3. **Construction of new frames.** Our construction of frames for the Triebel-Lizorkin and Besov spaces on the sphere relies on the general approach from Theorem 2.7.

Suppose $\hat{a}$ is the function from the definition of needlets in (4.7) and let us denote again by $\hat{a}$ its even extension to $\mathbb{R}$, i.e. $\hat{a}(-t) = \hat{a}(t)$. The inverse Fourier transform a of $\hat{a}$ is then real valued, even, and belongs to the Schwartz class $\mathcal{S}$ of rapidly decaying functions on $\mathbb{R}$. For given $M > 1$, an integer $N \ge 1$, and $\varepsilon > 0$, we construct an even function $b \in C^\infty(\mathbb{R})$ obeying the following conditions:

$$\begin{aligned} &(i) && \operatorname{supp} b \subset [-R, R] \ \text{ for some } R > 0,\\ &(ii) && |a^{(r)}(t) - b^{(r)}(t)| \le \varepsilon(1 + |t|)^{-M} \quad \text{for } 0 \le r \le N + n - 1,\\ &(iii) && \int_{\mathbb{R}} t^r b(t)\, dt = 0 \quad \text{for } 0 \le r \le N + n - 2. \end{aligned} \tag{4.20}$$

Note that the Fourier transform $\hat{b}$ of b is even and belongs to $\mathcal{S}$. A scheme for constructing this sort of functions b will be given below.

Just as in the construction of needlets we shall use $\mathcal{X} = \cup_{j=j_0}^\infty \mathcal{X}_j$ (see (4.13)) as an index set as well as a set of localization points for the new elements. For each $\xi \in \mathcal{X}_j$ $(j \ge j_0)$ we define the function θ_ξ on the sphere by

$$\theta_\xi(x) := c_\xi^{1/2} \sum_{\nu=0}^\infty \hat{b}\Big(\frac{\nu + \lambda}{2^j}\Big) \mathsf{P}_\nu(\xi \cdot x), \quad \lambda := (n-1)/2, \tag{4.21}$$

and then $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$ is our new system on $\mathbb{S}^n$.

With the next theorems we show that for appropriately selected parameters M, N, and ε, Θ is a frame for the F- and B- spaces with the claimed support property.

Let $\mathcal{J} := n/\min\{1, p, q\}$ in the case of F-spaces and $\mathcal{J} := n/\min\{1, p\}$ for B-spaces.

Theorem 4.7. *Suppose* $s \in \mathbb{R}$, $0 < p, q < \infty$ *and let* $\Theta := \{\theta_\xi\}_{\xi\in\mathcal{X}}$ *be constructed as above with* b *satisfying* (4.20), *where* $M > \mathcal{J}$ *and* $N > \max\{s, \mathcal{J} - n - s, 1\}$. *Then for sufficiently small* $\varepsilon > 0$ *the system* Θ *is a frame for the spaces* $L^2(\mathbb{S}^n)$, F_p^{sq}, *and* B_p^{sq} *in the sense of Definition 2.5. In particular, we have:*

(a) The operator

$$Sf := \sum_{\xi\in\mathcal{X}} \langle f, \theta_\xi\rangle \theta_\xi, \tag{4.22}$$

where $\langle f, \theta_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle f, \psi_\eta\rangle\langle\psi_\eta, \theta_\xi\rangle$*, is bounded and invertible on* $L^2(\mathbb{S}^n)$*,* F_p^{sq}*,* B_p^{sq}*, and* S^{-1} *is also bounded on* $L^2(\mathbb{S}^n)$*,* F_p^{sq}*,* B_p^{sq}*, and*

$$S^{-1}f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle S^{-1}\theta_\xi. \tag{4.23}$$

(b) If $f \in \mathcal{S}'$*, then* $f \in F_p^{sq}$ *if and only if* $(\langle f, S^{-1}\theta_\xi\rangle) \in f_p^{sq}$*, and for* $f \in F_p^{sq}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \text{and} \quad \|f\|_{F_p^{sq}} \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{f_p^{sq}}. \tag{4.24}$$

(c) If $f \in \mathcal{S}'$*, then* $f \in B_p^{sq}$ *if and only if* $(\langle f, S^{-1}\theta_\xi\rangle) \in b_p^{sq}$*, and for* $f \in B_p^{sq}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, S^{-1}\theta_\xi\rangle\theta_\xi \quad \text{and} \quad \|f\|_{B_p^{sq}} \sim \|(\langle f, S^{-1}\theta_\xi\rangle)\|_{b_p^{sq}}. \tag{4.25}$$

The convergence in (4.22)-(4.25) *is unconditional in the respective space* L^2*,* F_p^{sq}*, or* B_p^{sq}*. Above, (b) and (c) also hold with the roles of* θ_ξ *and* $S^{-1}\theta_\xi$ *interchanged.*

Moreover, for any $\xi \in \mathcal{X}_j$*,* $j \ge j_0$*, the element* θ_ξ *is supported on the spherical cap* $B_\xi(R2^{-j})$*, where* $R > 0$ *is the constant from* (4.20).

Several remarks are in order:

(a) Atomic decompositions are available for various spaces and in particular for Triebel-Lizorkin and Besov spaces on $\mathbb{R}^n$ (see [6]). Theorem 4.7 provides atomic decompositions for Triebel-Lizorkin and Besov spaces on $\mathbb{S}^n$. These atomic decompositions have the advantage that they involve atoms from a fixed sequence Θ, while in general the atoms in the atomic decompositions may vary with the distributions.

(b) Note that the function $b \in C^\infty$ from our construction is not necessarily compactly supported. As long as b satisfies conditions (ii)-(iii) in (4.20) it will induce a frame for the F- and B-spaces on $\mathbb{S}^n$. In addition to this the nature of b or $\hat{b}$ can be prescribed, e.g. b or $\hat{b}$ can be a low degree rational function or a linear combination of a small number of dilations and shifts of the Gaussian e^{-t^2}.

(c) We would like to point out that the elements of Θ are essentially rotations and spectral dilations of a single function supported on a cap on the sphere and hence bear some resemblance with compactly supported wavelets.

We start with the construction of a function b obeying (4.20). Then we shall carry out the proof of Theorem 4.7 in several steps. The gist of the proof will be the interplay between the spherical harmonics and the classical Fourier transform related by the Dirichlet-Mehler representation of Gegenbauer polynomials.

4.4. **Construction of b.** A first step in constructing the frame $\{\theta_\xi\}$ is the construction of a function b satisfying conditions (4.20), which we give in the next theorem. As will be seen this construction allows to prescribe the nature of b or $\hat{b}$.

Theorem 4.8. *For given* $M > 0$*,* $N \ge 1$*, and* $\varepsilon > 0$*, here exists an even real valued function* $b \in C^\infty$ *which satisfies conditions* (4.20).

Proof. The construction of a function b with the claimed properties follows the same lines as in the proof of Theorem 4.1 in [13]. Therefore, we shall only outline the main steps in this construction.

We pick an even function $\phi \in C^\infty$such that $\operatorname{supp} \phi \subset [-1,1]$ and $\int_{\mathbb{R}} \phi = 1$. Write $\phi_k(t) := k\phi(kt)$ and denote by Φ_k the set of all finite linear combinations of shifts of ϕ_k, i.e. functions g of the form $g(t) = \sum_j a_j\phi_k(t+b_j)$, where the sum is finite.

We first show that for every $\varepsilon > 0$ and an even (or odd) function $h \in C^\infty$ there exist $k > 0$ (sufficiently large) and an even (or odd) function $g \in \Phi_k$ such that

$$|h^{(r)}(t) - g^{(r)}(t)| \le \varepsilon(1+|t|)^M, \quad t \in \mathbb{R}, \ \ r = 0,1,\dots,N_0, \tag{4.26}$$

where $N_0 := N + n - 1$. Indeed, define $g_k := h * \phi_k$. Since $\int_{\mathbb{R}} \phi_k = 1$, then

$$h^{(r)}(t) - g_k^{(r)}(t) = \int_{\mathbb{R}} [h^{(r)}(t) - h^{(r)}(t-y)]\phi_k(y)dy$$

and taking k sufficiently large one easily shows that

$$|h^{(r)}(t) - g_k^{(r)}(t)| \le (\varepsilon/2)(1+|t|)^{-M}, \quad t \in \mathbb{R}, \ \ r = 0,1,\dots,N_0. \tag{4.27}$$

Notice that g_k is even (odd) if h is even (odd).

To discretize the approximant g_k we first observe that since $h \in \mathcal{S}$, there exists $R > 0$ such that

$$|h^{(r)}(t)| \le \varepsilon(1+|t|)^{-M}, \quad |t| \ge R, \ \ r = 0,1,\dots,N_0. \tag{4.28}$$

Now, we choose sufficiently large $S > 0$ so that $J := SR$ is an integer and consider the points $t_j := \frac{j-1/2}{S}$, $j = 1,\dots,J$, and $t_j := \frac{j+1/2}{S}$, $j = -1,\dots,-J$. We define

$$g(t) := S^{-1} \sum_{-J \le j \le J, j \ne 0} h(t_j)\phi_k(t - t_j),$$

which can be viewed as a Riemann sum for the integral $\int_R^R h(y)\phi_k(t-y)dy$. Notice that $g_k(t) = \int_{\mathbb{R}} h(y)\phi_k(t-y)dy$. As in the proof of Theorem 4.1 in [13], one easily shows, using (4.27)-(4.28), that for sufficiently large S this function satisfies (4.26). In addition to this, evidently g is even (odd) if h is even (odd) and $g \in \Phi_k$.

Our second step is to utilize the result of the first step to construct the desired function b. Consider the shift operator $T_\delta f(t) := f(t+\delta)$. Then the sth centered difference is defined by $\Delta_\delta^s f := (T_\delta - T_{-\delta})^s f$ and it is easy to see that its Fourier transform satisfies $(\Delta_\delta^s f)^\wedge(\xi) = (2i\sin\delta\xi)^s\hat{f}(\xi)$.

We choose $s := N_0$ and $0 < \delta \le 1/s$, and define the function h from the identity $\hat{h}(\xi) := \frac{\hat{a}(\xi)}{(2i\sin\delta\xi)^s}$. Since $\hat{a}(\xi) = 0$ for $\xi \in [-1/2, 1/2]$, then $\hat{h} \in \mathcal{S}$ and hence $h \in \mathcal{S}$. Further, since $\hat{a}$ is even, then $\hat{h}$ and h are even (odd) if s is even (odd). Moreover, by the construction $a = \Delta_\delta^s h$. We now use the result of the first step to construct a function $g \in \Phi_k$ such that g satisfies (4.26) with h from above.

After this preparation, we define $b := \Delta_\delta^s g$ and claim that b has the desired properties. Indeed, note that $a^{(r)} - b^{(r)} = \Delta_\delta^s(h^{(r)} - g^{(r)})$ and by (4.26) we infer

$$|a^{(r)}(t) - b^{(r)}(t)| \le \varepsilon 2^{s+M}(1+|t|)^{-M}, \quad r = 0,1,\dots,N_0. \tag{4.29}$$

On the other hand

$$\int_{\mathbb{R}} t^r b(t)dt = \int_{\mathbb{R}} t^r \Delta_\delta^s g(t)dt = (-1)^s \int_{\mathbb{R}} g(t)\Delta_\delta^s t^r dt = 0, \quad r = 0,1,\dots,s-1.$$

Also, note that $b := \Delta_\delta^s g$ is even if g and s are both odd or even and evidently $b \in \Phi_k$ and hence b is compactly supported. We finally observe that since ε is independent of M and s the factor $\varepsilon 2^{s+M}$ in (4.29) can be replaced by ε. $\square$

4.5. **Almost diagonal matrices.** To show that the new system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\}$ is a frame for Triebel-Lizorkin and Besov spaces we shall use Theorem 2.7 with $L := F_p^{sq}(\mathbb{S}^n)$ or $B_p^{sq}(\mathbb{S}^n)$ and $\ell(\mathcal{X}) := f_p^{sq}(\mathcal{X})$ or $b_p^{sq}(\mathcal{X})$, respectively. Then $L^2(\mathbb{S}^n)$ is the natural selection of an associated Hilbert space. By Theorem 2.7 it readily follows that Θ is a frame for F_p^{sq} (or B_p^{sq}) if the operators with matrices

$$\begin{aligned}
&\mathbf{A} := (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \qquad && a_{\xi,\eta} := \langle \psi_\eta, \psi_\xi \rangle,\\
&\mathbf{B} := (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, && b_{\xi,\eta} := \langle \theta_\eta, \psi_\xi \rangle,\\
&\mathbf{C} := (c_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, && c_{\xi,\eta} := \langle \psi_\eta, \theta_\xi \rangle\\
&\mathbf{D} := (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, && d_{\xi,\eta} := \langle \psi_\eta, \psi_\xi - \theta_\xi \rangle,\\
&\mathbf{E} := (e_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, && e_{\xi,\eta} := \langle \psi_\eta - \theta_\eta, \psi_\xi \rangle,
\end{aligned} \tag{4.30}$$

are bounded on f_p^{sq} (or b_p^{sq}), and $\|\mathbf{D}\|_{f_p^{sq}\mapsto f_p^{sq}} \le \varepsilon$, $\|\mathbf{E}\|_{f_p^{sq}\mapsto f_p^{sq}} \le \varepsilon$ (respectively, $\|\mathbf{D}\|_{b_p^{sq}\mapsto b_p^{sq}} \le \varepsilon$, $\|\mathbf{E}\|_{b_p^{sq}\mapsto b_p^{sq}} \le \varepsilon$) for sufficiently small ε.

In analogy with the classical case on $\mathbb{R}^n$ (see [6]), we shall show the boundedness of the above operators by using the machinery of the almost diagonal operators.

It will be convenient to us to denote

$$\ell(\xi) := 2^{-j} \quad \text{for} \quad \xi \in \mathcal{X}_j,\ j \ge j_0. \tag{4.31}$$

Evidently, $\ell(\xi)$ is a constant multiple of the radius of the cap G_ξ.

Definition 4.9. *Let $\mathbf{A}$ be a linear operator acting on $f_p^{sq}(\mathcal{X})$ or $b_p^{sq}(\mathcal{X})$ with associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$. We say that $\mathbf{A}$ is almost diagonal if there exists $\delta > 0$ such that*

$$\sup_{\xi,\eta\in\mathcal{X}} \frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)} < \infty,$$

where

$$\begin{aligned}
\omega_\delta(\xi,\eta) \quad := \quad & \left(\frac{\ell(\xi)}{\ell(\eta)}\right)^s \left(1 + \frac{d(\xi,\eta)}{\max\{\ell(\xi),\ell(\eta)\}}\right)^{-\mathcal{J}-\delta}\\
\times \quad & \min\left\{ \left(\frac{\ell(\xi)}{\ell(\eta)}\right)^{(n+\delta)/2}, \left(\frac{\ell(\eta)}{\ell(\xi)}\right)^{(n+\delta)/2+(\mathcal{J}-n)} \right\},
\end{aligned}$$

with $\mathcal{J} := n/\min\{1,p,q\}$ for f_p^{sq} and $\mathcal{J} := n/\min\{1,p\}$ for b_p^{sq}.

The almost diagonal operators are bounded on f_p^{sq} and b_p^{sq}. More precisely, with the notation

$$\|\mathbf{A}\|_\delta := \sup_{\xi,\eta\in\mathcal{X}} \frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)} \tag{4.32}$$

the following result holds:

Theorem 4.10. *Suppose $s \in \mathbb{R}$, $0 < q \le \infty$, and $0 < p < \infty$ ($0 < p \le \infty$ in the case of b-spaces) and let $\|\mathbf{A}\|_\delta < \infty$ (in the sense of Definition 4.9) for some $\delta > 0$. Then there exists a constant $c > 0$ such that for any sequence $h := \{h_\xi\}_{\xi\in\mathcal{X}} \in f_p^{sq}$*

$$\|\mathbf{A}h\|_{f_p^{sq}} \le c\|\mathbf{A}\|_\delta \|h\|_{f_p^{sq}}, \tag{4.33}$$

and for any sequence $h := \{h_\xi\}_{\xi\in\mathcal{X}} \in b_p^{sq}$

$$\|\mathbf{A}h\|_{b_p^{sq}} \le c\|\mathbf{A}\|_\delta \|h\|_{b_p^{sq}}. \tag{4.34}$$

The proof of this theorem is quite similar to the proof of Theorem 3.3 in [6]. For completeness we give it in the appendix.

The above theorem indicates that to prove that Θ is a frame for F_p^{sq} (or B_p^{sq}) it suffices to show that the operators with matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$, defined in (4.30) , are almost diagonal and

$$\|\mathbf{D}\|_\delta \le \varepsilon, \quad \|\mathbf{E}\|_\delta \le \varepsilon \tag{4.35}$$

for a fixed $\delta > 0$ and sufficiently small $\varepsilon > 0$.

4.6. **Representation and localization of kernels. Estimation of supp $\boldsymbol{\theta_\xi}$.** Kernels of the form

$$\Lambda_N(\xi\cdot\eta) := \sum_{\nu\ge 0} \hat{g}\Big(\frac{\nu+\lambda}{N}\Big)\mathsf{P}_\nu(\xi\cdot\eta), \quad \xi,\eta\in\mathbb{S}^n, \ \ N\ge 1, \tag{4.36}$$

will play an important role in the proof of Theorem 4.7. Here as everywhere else P_ν and λ are from (4.1).

Lemma 4.11. *For an even function* $\hat{g} \in \mathcal{S}$ *the kernel* Λ_N *from above has the representation*

$$\Lambda_N(\cos\alpha) = \frac{c_n}{(\sin\alpha)^{n-2}} \int_\alpha^\pi (\cos\alpha - \cos\varphi)^{\lambda-1} K_N(\varphi)d\varphi, \quad 0\le\alpha\le\pi, \tag{4.37}$$

where

$$K_N(\alpha) = (\pi/2)N\sum_{\nu\in\mathbb{Z}} (-1)^{\nu(n-1)} R_n\Big(\frac{d}{d\alpha}\Big) g(N(\alpha+2\pi\nu)) \tag{4.38}$$

with

$$R_n(z) := \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} (-z^2-(\lambda-r)^2) \times \begin{cases} -z\sin\lambda\pi, & n \ \textit{even} \\ \cos\lambda\pi, & n \ \textit{odd} \end{cases} \tag{4.39}$$

and $c_n > 0$ *depends only on* n.

This lemma is in essence contained in [19], see Proposition 3.2. For completeness we give its proof in the appendix.

We next give an estimate of the localization of the kernels Λ_N from (4.36) provided g and its derivatives are well localized.

Lemma 4.12. *If* $g \in C^{n-1}(\mathbb{R})$ *is even and*

$$|g^{(m)}(t)| \le \frac{A}{(1+|t|)^M}, \quad t\in\mathbb{R},\ 0\le m\le n-1, \tag{4.40}$$

for some constants $M > 1$ *and* $A > 0$*, then*

$$|\Lambda_N(\cos\alpha)| \le \frac{cAN^n}{(1+N\alpha)^M}, \quad 0\le\alpha\le\pi, \tag{4.41}$$

where $c > 0$ *depends only on* M *and* n.

Proof. We use (4.40) and that $R_n(z)$ from (4.39) is a polynomial of degree $n-1$ to obtain

$$|K_N(\alpha)| \le cAN \sum_{\nu\in\mathbb{Z}} \frac{N^{n-1}}{(1+N|\alpha+2\pi\nu|)^M} \le \frac{cAN^n}{(1+N\alpha)^M}.$$

Now, precisely as in [19, §3.4] one shows that the above estimate used in (4.37) yields (4.41). We skip the details. □

Lemma 4.13. *For every $\xi \in \mathcal{X}_j$, $j \ge j_0$, θ_ξ is supported on the spherical cap of radius $R2^{-j}$ centered at ξ, where R is from (4.20), (i).*

Proof. Let $\xi \in \mathcal{X}_j$, $j \ge j_0$. Then by the definition of θ_ξ in (4.21) along with Lemma 4.11, we have

$$\theta_\xi(x) = \frac{c_n}{(\sin\phi)^{n-2}} \int_\phi^\pi (\cos\phi - \cos\varphi)^{\lambda-1} K_j(\varphi) d\varphi, \quad \xi\cdot x =: \cos\phi, \tag{4.42}$$

where

$$K_j(\varphi) := (\pi/2) c_\xi^{1/2} 2^j \sum_{\nu\in\mathbb{Z}} (-1)^{\nu(n-1)} R_n\Big(\frac{d}{d\varphi}\Big) b(2^j(\varphi+2\pi\nu)).$$

By construction $\operatorname{supp} b \subset [-R,R]$ and, hence, $\operatorname{supp} K_j \subset [-R2^{-j}, R2^{-j}]$ whenever $R2^{-j} < \pi$. This and (4.42) apparently lead to $\operatorname{supp}\theta_\xi \subset B_\xi(R2^{-j})$. The case when $R2^{-j} \ge \pi$ is trivial. □

4.7. **Estimation of inner products.** We shall need an estimate on the localization of the convolution of two well localized functions. In the following, for a given function g on $\mathbb{R}$ we denote $g_j(t) := 2^j g(2^j t)$.

Lemma 4.14. *Suppose the functions $g \in C^N(\mathbb{R})$ and $h \in C(\mathbb{R})$ satisfy the conditions:*

$$|g^{(r)}(t)| \le \frac{A_1}{(1+|t|)^{M_1}}, \quad 0 \le r \le N, \quad |h(t)| \le \frac{A_2}{(1+|t|)^{M_2}},$$

and

$$\int_{\mathbb{R}} t^r h(t) dt = 0 \quad \text{for } 0 \le r \le N-1,$$

where $N \ge 1$, $M_2 \ge M_1$, $M_2 > N+1$, and $A_1, A_2 > 0$. Then for $k \ge j$

$$|g_j * h_k(t)| \le cA_1A_2 2^{-(k-j)N} \frac{2^j}{(1+2^j|t|)^{M_1}},$$

where $c > 0$ depends only on M_1, M_2, and N.

The proof of this lemma is almost identical to the proof of Lemma B.1 in [6] and will be omitted. The only difference is in the normalization of the functions.

We now come to the main lemma which will enable us to estimate the inner products involved in (4.30). For simplicity, in the following we assume that $g, h \in \mathcal{S}$. Then their Fourier transforms $\hat g, \hat h \in \mathcal{S}$ as well, with $\mathcal{S}$ being the Schwartz class. For $\xi \in \mathcal{X}_j, j \ge j_0$, and $\eta \in \mathcal{X}_k, k \ge j_0$, we define

$$G_\xi(x) := c_\xi^{1/2} \sum_{\nu=0}^\infty \hat g\Big(\frac{\nu+\lambda}{2^j}\Big) \mathsf{P}_\nu(\xi\cdot x), \quad H_\eta(x) := c_\eta^{1/2} \sum_{\nu=0}^\infty \hat h\Big(\frac{\nu+\lambda}{2^k}\Big) \mathsf{P}_\nu(\eta\cdot x), \tag{4.43}$$

where c_ξ, c_η are from (4.11).

Lemma 4.15. *Suppose $g, h \in \mathcal{S}$ are both even and real valued,*

$$|g^{(m)}(t)| \le \frac{A_1}{(1+|t|)^M} \quad \text{and} \quad |h^{(m)}(t)| \le \frac{A_2}{(1+|t|)^M}, \quad 0 \le m \le N+n-1, \tag{4.44}$$

and

$$\int_{\mathbb{R}} t^r g(t)dt = \int_{\mathbb{R}} t^r h(t)dt = 0, \quad 0 \le m \le N+n-2, \tag{4.45}$$

where $N > 1$ and $M > N+1$. Then for $\xi \in \mathcal{X}_j$ and $\eta \in \mathcal{X}_k$

$$|\langle G_\xi, H_\eta\rangle| \le cA_1A_2 2^{-|k-j|(N+n/2)} \Big(1 + 2^{\min\{j,k\}} d(\xi, \eta)\Big)^{-M} \tag{4.46}$$

where $c > 0$ depends only on N, M, and n.

Proof. Assume that $k \ge j$ and let $\xi \cdot \eta =: \cos\alpha$, $0 \le \alpha \le \pi$. Then using that

$$\int_{\mathbb{S}^n} \mathsf{P}_\nu(\xi\cdot x)\mathsf{P}_\ell(\xi \cdot x)dx = \delta_{\nu,\ell}\mathsf{P}_\nu(\xi\cdot\eta),$$

$c_\xi \sim 2^{-jn}$ if $\xi \in \mathcal{X}_j$, and $c_\eta \sim 2^{-kn}$ if $\eta \in \mathcal{X}_k$, we have

$$\langle G_\xi, H_\eta\rangle \sim 2^{-(k+j)\frac{n}{2}} \sum_{\nu=0}^{\infty} \hat{g}\Big(\frac{\nu+\lambda}{2^j}\Big)\hat{h}\Big(\frac{\nu+\lambda}{2^k}\Big)\mathsf{P}_\nu(\xi\cdot\eta).$$

It is easy to see that

$$\hat{g}\Big(\frac{\nu+\lambda}{2^j}\Big)\hat{h}\Big(\frac{\nu+\lambda}{2^k}\Big) = (g_j * h_k)^\wedge(\nu+\lambda) = (g * h_{k-j})^\wedge\Big(\frac{\nu+\lambda}{2^j}\Big).$$

On the other hand,

$$(g * h_{k-j})^{(m)}(t) = (g^{(m)} * h_{k-j})(t)$$

and therefore, by Lemma 4.14,

$$|(g*h_{k-j})^{(m)}(t)| \le \frac{cA_1A_2 2^{-(k-j)N}}{(1+|t|)^M}, \quad 0 \le m \le n-1.$$

We now invoke Lemma 4.12 to obtain

$$|\langle G_\xi, H_\eta\rangle| \le cA_1A_2 2^{-(k+j)\frac{n}{2}} 2^{-(k-j)N} \frac{2^{jn}}{(1+2^j\alpha)^M} \le cA_1A_2 \frac{2^{-(k-j)(N+\frac{n}{2})}}{(1+2^j\alpha)^M}. \qquad \square$$

Proof of Theorem 4.7. Evidently, Theorem 4.7 will follow by Theorem 2.7, applied with $H := L^2(\mathbb{S}^n)$, $L := F_p^{sq}$ and $\ell(\mathcal{X}) := f_p^{sq}$ (or $L := B_p^{sq}$ and $\ell(\mathcal{X}) := b_p^{sq}$), and Ψ the frame from Theorem 4.5, if we prove that the matrices defined in (4.30) are almost diagonal and $\|\mathbf{D}\|_\delta < \varepsilon$, $\|\mathbf{E}\|_\delta < \varepsilon$ for some $\delta > 0$ and sufficiently small ε (see (2.31)).

Here, we only give the argument regarding the estimate $\|\mathbf{D}\|_\delta < \varepsilon$; the proof of the estimate $\|\mathbf{E}\|_\delta < \varepsilon$ is the same. By the definition of the needlet ψ_ξ for $\xi \in \mathcal{X}_j$ ($j \ge j_0$) we have

$$\psi_\xi(x) := c_\xi^{1/2} \sum_{\nu=0}^{\infty} \hat{a}\Big(\frac{\nu+\lambda}{2^j}\Big)\mathsf{P}_\nu(\xi\cdot x).$$

Since $\hat{a} \in C^\infty$ is compactly supported and $\hat{a}(t) = 0$ for $t \in [-1/2, 1/2]$, there exists a constant $A_1 > 0$ such that

$$|a^{(r)}(t)| \le A_1(1+|t|)^{-M}, \quad 0 \le r \le N+n-1, \quad \text{and} \quad \int_{\mathbb{R}} t^r a(t)dt = 0, \quad r \ge 0.$$

On the other hand, from the definition of θ_ξ in (4.21) it follows that

$$\psi_\eta(x) - \theta_\eta(x) = c_\eta^{1/2} \sum_{\nu=0}^{\infty} (a-b)^\wedge\Big(\frac{\nu+\lambda}{2^k}\Big) \mathsf{P}_\nu(\eta\cdot x), \quad \eta \in \mathcal{X}_k,$$

and from the construction of b we have

$$|(a-b)^{(r)}(t)| \le \varepsilon(1+|t|)^{-M}, \quad 0 \le r \le N+n-1, \quad \text{and}$$
$$\int_{\mathbb{R}} t^r (a-b)(t)dt = 0, \quad 0 \le r \le N+n-2.$$

We now apply Lemma 4.15 with $g = a$ and $h := a-b$ to obtain

$$|\langle \psi_\xi, \psi_\eta - \theta_\eta\rangle| \le cA_1\varepsilon \min\Big\{\frac{\ell(\xi)}{\ell(\eta)}, \frac{\ell(\eta)}{\ell(\xi)}\Big\}^{N+\frac{n}{2}} \Big(1 + \frac{d(\xi,\eta)}{\max\{\ell(\xi),\ell(\eta)\}}\Big)^{-M}$$

and since $M > \mathcal{J}$ and $N > \max\{s, \mathcal{J}-n-s\}$, we get $\|\mathbf{D}\|_\delta < cA_1\varepsilon$. However, ε is independent of c, A_1, M, and N, therefore, $cA_1\varepsilon$ above can be replaced by ε. $\square$

5. Appendix

Proof of Theorem 4.10. We need the maximal operator on $\mathbb{S}^n$. Denote by $\mathcal{G}$ the set of all spherical caps on $\mathbb{S}^n$, i.e. $G \in \mathcal{G}$ if G is of the form: $G := \{x \in \mathbb{S}^n : d(x,\eta) < \rho\}$ with $\eta \in \mathbb{S}^n$ and $\rho > 0$. The maximal operator $\mathcal{M}_t$ $(t>0)$ is defined by

$$\mathcal{M}_t f(x) := \sup_{G\in\mathcal{G}:\, x\in G} \left(\frac{1}{|G|}\int_G |f(\omega)|^t\, d\omega\right)^{1/t}, \quad x \in \mathbb{S}^n.$$

We shall use the Fefferman-Stein vector-valued maximal inequality (see [26]): If $0 < p < \infty$, $0 < q \le \infty$, and $0 < t < \min\{p,q\}$, then for any sequence of functions $f_1, f_2, \dots$ on $\mathbb{S}^n$

$$\Big\|\Big(\sum_{j=1}^{\infty} [\mathcal{M}_t f_j(\cdot)]^q\Big)^{1/q}\Big\|_{L^p} \le c\Big\|\Big(\sum_{j=1}^{\infty} |f_j(\cdot)|^q\Big)^{1/q}\Big\|_{L^p} \tag{5.1}$$

where $c = c(p,q,t,n)$.

The next lemma will also be needed.

Lemma 5.1. *Let $0 < t \le 1$ and $M > d/t$. For any sequence of complex numbers $\{h_\eta\}_{\eta\in\mathcal{X}_m}$, $m \ge 0$, we have for $x \in G_\xi$, $\xi \in \mathcal{X}$,*

$$\sum_{\eta\in\mathcal{X}_m} |h_\eta|\Big(1+\frac{d(\xi,\eta)}{\max\{\ell(\xi),\ell(\eta)\}}\Big)^{-M} \le c\max\Big\{\Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^{\frac{d}{t}}, 1\Big\} M_t\Big(\sum_{\eta\in\mathcal{X}_m} |h_\eta|\mathbb{1}_\eta\Big)(x).$$

When $\ell(\xi) \le \ell(\eta)$, this lemma is Lemma 4.8 in [20]. The proof in the case $\ell(\xi) > \ell(\eta)$ is similar and will be omited (see also Remark A.3 in [6]).

We shall only prove estimate (4.33). The proof of (4.34) is similar and we omit it. Let $\mathbf{A}$ be an almost diagonal operator on f_p^{sq} with associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$ and let $h \in f_p^{sq}$. Then $(\mathbf{A}h)_\xi = \sum_{\eta\in\mathcal{X}} a_{\xi\eta}h_\eta$, where the series converges absolutely (see proof below). Then

$$\begin{aligned}\|\mathbf{A}h\|_{f_p^{sq}} &:= \Big\|\Big(\sum_{\xi\in\mathcal{X}} \big[|G_\xi|^{-s/n-1/2}|(\mathbf{A}h)_\xi|\mathbb{1}_{G_\xi}\big]^q\Big)^{1/q}\Big\|_{L^p} \\ &\le c\Big\|\Big(\sum_{\xi\in\mathcal{X}} \big[\ell(\xi)^{-s-n/2}\sum_{\eta\in\mathcal{X}} |a_{\xi\eta}||h_\eta|\mathbb{1}_{G_\xi}\big]^q\Big)^{1/q}\Big\|_{L^p} \le c(\Sigma_1+\Sigma_2),\end{aligned}$$

where

$$\Sigma_1 := \Big\| \Big(\sum_{\xi\in\mathcal{X}} \big[\ell(\xi)^{-s-n/2} \sum_{\ell(\eta)\le\ell(\xi)} |a_{\xi\eta}||h_\eta| \mathbb{1}_{G_\xi}\big]^q \Big)^{1/q} \Big\|_{L^p} \quad \text{and}$$

$$\Sigma_2 := \Big\| \Big(\sum_{\xi\in\mathcal{X}} \big[\ell(\xi)^{-s-n/2} \sum_{\ell(\eta)>\ell(\xi)} |a_{\xi\eta}||h_\eta| \mathbb{1}_{G_\xi}\big]^q \Big)^{1/q} \Big\|_{L^p}.$$

Since $\|\mathbf{A}\|_\delta < \infty$, we have whenever $\ell(\eta) \le \ell(\xi)$

$$|a_{\xi\eta}| \le c\|\mathbf{A}\|_\delta \Big(\frac{\ell(\eta)}{\ell(\xi)}\Big)^{\mathcal{J}-s-n/2+\delta/2} \Big(1+\frac{d(\xi,\eta)}{\ell(\xi)}\Big)^{-\mathcal{J}-\delta}.$$

Choose $0 < t < \min\{1, p, q\}$ so that $\mathcal{J} - d/t + \delta/2 > 0$. Let $\lambda_\xi := \ell(\xi)^{-s-n/2}\mathbb{1}_{G_\xi}$. Then we have

$$\frac{\Sigma_1}{\|\mathbf{A}\|_\delta} \le c\Big\| \Big(\sum_{\xi\in\mathcal{X}} \Big(\sum_{\ell(\eta)\le\ell(\xi)} \Big(\frac{\ell(\eta)}{\ell(\xi)}\Big)^{\mathcal{J}-s-\frac{n}{2}+\frac{\delta}{2}} \Big(1+\frac{d(\xi,\eta)}{\ell(\xi)}\Big)^{-\mathcal{J}-\delta} |h_\eta|\lambda_\xi\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}$$

$$= c\Big\| \Big(\sum_{j\ge 0}\sum_{\xi\in\mathcal{X}_j} \Big(\sum_{m\ge j} 2^{(j-m)(\mathcal{J}-s-\frac{n}{2}+\frac{\delta}{2})} \sum_{\eta\in\mathcal{X}_m} \big(1+2^j d(\xi,\eta)\big)^{-\mathcal{J}-\delta} |h_\eta|\lambda_\xi\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}.$$

We now apply Lemma 5.1 and the maximal inequality (5.1) to obtain

$$\frac{\Sigma_1}{\|\mathbf{A}\|_\delta} \le c\Big\| \Big(\sum_{j\ge 0}\sum_{\xi\in\mathcal{X}_j} \Big(\sum_{m\ge j} 2^{(j-m)(\mathcal{J}-s-\frac{n}{2}+\frac{\delta}{2}-\frac{n}{t})} M_t\big(\sum_{\eta\in\mathcal{X}_m} |h_\eta|\chi_\eta\big)\lambda_\xi\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}$$

$$\le c\Big\| \Big(\sum_{j\ge 0} \Big(\sum_{m\ge j} 2^{(j-m)(\mathcal{J}-\frac{n}{t}+\frac{\delta}{2})} M_t\big(\sum_{\eta\in\mathcal{X}_m} |h_\eta|\lambda_\eta\big)\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}$$

$$\le c\Big\| \Big(\sum_{j\ge 0} \Big(M_t\big(\sum_{\xi\in\mathcal{X}_j} |h_\xi|\lambda_\xi\big)\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p} \le c\|h\|_{f_p^{sq}}.$$

If $\ell(\eta) > \ell(\xi)$, then

$$|a_{\xi\eta}| \le c\|\mathbf{A}\|_\delta \Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^{s+d/2+\delta/2} \Big(1+\frac{d(\xi,\eta)}{\ell(\eta)}\Big)^{-\mathcal{J}-\delta}$$

and hence

$$\frac{\Sigma_2}{\|\mathbf{A}\|_\delta} \le c\Big\| \Big(\sum_{\xi\in\mathcal{X}} \Big(\sum_{\ell(\eta)>\ell(\xi)} \Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^{s+\frac{n}{2}+\frac{\delta}{2}} \Big(1+\frac{d(\xi,\eta)}{\ell(\eta)}\Big)^{-\mathcal{J}-\delta} |h_\eta|\lambda_\xi\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}$$

$$= c\Big\| \Big(\sum_{j\ge 0}\sum_{\xi\in\mathcal{X}_n} \Big(\sum_{m<j} 2^{(m-j)(s+\frac{n}{2}+\frac{\delta}{2})} \sum_{\eta\in\mathcal{X}_m} \big(1+2^m d(\xi,\eta)\big)^{-\mathcal{J}-\delta} |h_\eta|\lambda_\xi\Big)^q\Big)^{\frac{1}{q}} \Big\|_{L^p}.$$

Employing again Lemma 5.1 and the maximal inequality (5.1) we obtain

$$\frac{\Sigma_2}{\|\mathbf{A}\|_\delta} \le c\Big\|\Big(\sum_{j\ge 0}\sum_{\xi\in\mathcal{X}_j}\Big(\sum_{m<j}2^{(m-j)(s+\frac{n}{2}+\frac{\delta}{2})}M_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\chi_\eta\big)\lambda_\xi\Big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p}$$

$$\le c\Big\|\Big(\sum_{j\ge 0}\Big(\sum_{m<j}2^{(m-j)(\delta/2)}M_t\big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\lambda_\eta\big)\Big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p}$$

$$\le c\Big\|\Big(\sum_{j\ge 0}\Big(M_t\big(\sum_{\xi\in\mathcal{X}_j}|h_\xi|\lambda_\xi\big)(x)\Big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \le c\|h\|_{f_p^{sq}}.$$

The above estimates for Σ_1 and Σ_2 yield (4.33). $\square$

Proof of Lemma 4.11. Recall first the Dirichlet-Mehler integral representation of Gegenbauer polynomials [3, p. 177]:

$$P_\nu^{(\lambda)}(\cos\alpha) = \frac{2^\lambda\Gamma(\lambda+\frac{1}{2})\Gamma(\nu+2\lambda)(\sin\alpha)^{1-2\lambda}}{\sqrt{\pi}\nu!\Gamma(\lambda)\Gamma(2\lambda)}\int_\alpha^\pi \frac{\cos\big((\nu+\lambda)\varphi-\lambda\pi\big)}{(\cos\alpha-\cos\varphi)^{1-\lambda}}\,d\varphi.$$

On account of (4.1), then (4.37) holds with

$$K_N(\alpha) = \sum_{\nu=0}^\infty \hat{g}\Big(\frac{\nu+\lambda}{N}\Big)\frac{(\nu+\lambda)(\nu+n-2)!}{\nu!}\times\begin{cases}\sin\lambda\pi\sin(\nu+\lambda)\alpha, & n \text{ even}\\ \cos\lambda\pi\cos(\nu+\lambda)\alpha, & n \text{ odd.}\end{cases}$$

Evidently, $\frac{(\nu+\lambda)(\nu+n-2)!}{\nu!} = (\nu+\lambda)(\nu+n-2)\dots(\nu+1)$ and a little algebra shows that

$$\frac{(\nu+\lambda)(\nu+n-2)!}{\nu!} = \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor}\big((\nu+\lambda)^2-(\lambda-r)^2\big)\times\begin{cases}\nu+\lambda, & n \text{ even}\\ 1, & n \text{ odd.}\end{cases}$$

Let now $Q_n(z)$ be the degree $n-1$ polynomial

$$Q_n(z) := \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor}\big(z^2-(\lambda-r)^2\big)\times\begin{cases}z\sin\lambda\pi, & n \text{ even}\\ \cos\lambda\pi, & n \text{ odd.}\end{cases}$$

Then

$$K_N(\alpha) = \sum_{\nu=0}^\infty \hat{g}\Big(\frac{\nu+\lambda}{N}\Big)Q_n(\nu+\lambda)\times\begin{cases}\sin(\nu+\lambda)\alpha, & n \text{ even}\\ \cos(\nu+\lambda)\alpha, & n \text{ odd.}\end{cases}$$

Note that $Q_n(-z) = (-1)^{n-1}Q_n(z)$ and Q_n has zeros $\pm(\lambda-r)$, $r = 1,\dots,\lfloor\frac{n-1}{2}\rfloor$. The critical step now is that since $\hat{g}$ is even and because of the symmetry and zeros of Q_n

$$K_N(\alpha) = (1/2)\sum_{\nu\in\mathbb{Z}}\hat{g}\Big(\frac{\nu+\lambda}{N}\Big)Q_n(\nu+\lambda)\times\begin{cases}\sin(\nu+\lambda)\alpha, & n \text{ even}\\ \cos(\nu+\lambda)\alpha, & n \text{ odd.}\end{cases} \tag{5.2}$$

Let

$$R_n(z) := \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor}\big(-z^2-(\lambda-r)^2\big)\times\begin{cases}-z\sin\lambda\pi, & n \text{ even}\\ \cos\lambda\pi, & n \text{ odd}\end{cases}$$

which is a polynomial of degree $n-1$ (related to Q_n). Then (5.2) can be rewritten in the form

$$K_N(\alpha) = (1/2)R_n\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}}\hat{g}\Big(\frac{\nu+\lambda}{N}\Big)\cos(\nu+\lambda)\alpha \tag{5.3}$$

$$= (1/4)R_n\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}}\hat{g}\Big(\frac{\nu+\lambda}{N}\Big)e^{(\nu+\lambda)\alpha}.$$

Here we again used that the part of the sum in (5.2) with indices $-(n-1)<\nu<0$ is void.

Recall the Poisson summation formula:

$$\sum_{\nu\in\mathbb{Z}} f(2\pi\nu) = (2\pi)^{-1}\sum_{\nu\in\mathbb{Z}}\hat{f}(\nu), \quad \text{where} \quad \hat{f}(\xi) := \int_{\mathbb{R}} f(y)e^{-i\xi y}dy,$$

and set $\hat{f}(\xi) := \hat{g}(\frac{\xi+\lambda}{N})e^{i(\xi+\lambda)t}$. Then $f(y) = Ne^{-i\lambda y}a\left(N(y+t)\right)$ and (5.3) along with the summation formula give

$$K_N(\alpha) = (\pi/2)NR_n\Big(\frac{d}{d\alpha}\Big)\sum_{\nu\in\mathbb{Z}} e^{-2\pi i\nu\lambda}g\left(N(\alpha+2\pi\nu)\right),$$

which implies (4.38). □

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