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A Posteriori Error Estimates for Finite Volume Approximation of Elliptic Equations on General Surfaces

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Abstract

In this paper, we present a residual-based a posteriori error estimate for the finite volume discretization of steady diffusion-convection-reaction equations defined on general surfaces in $\mathbb{R}^3$, which are often implicitly represented as level sets of smooth functions. Reliability and efficiency of the proposed a posteriori error estimator are rigorously proved. Numerical experiments are also conducted to verify the theoretical results and demonstrate the robustness of the error estimator.

Keywords: A posteriori error estimates, finite volume approximation, PDEs on surfaces

1 Introduction

In the area of numerical solution of partial differential equations (PDEs), *a posteriori* error estimators are computable quantities in terms of the approximate solutions, and provide a *reliable* and *efficient* measurement for the errors of the discrete solution without knowing the exact solution. Reliability is often referred to that the true error can be bounded from above by the error estimator and efficiency implies that the true error is also locally bounded from below by the error estimator. A posteriori error estimates have played a very important role in adaptive meshes generation and algorithm designing for numerical PDEs. Theoretical and systematical studies of a posteriori error estimators for finite element approximation began in the late of 1970s [5], and since then a vast number of literatures gradually appeared, see [2, 4, 6, 7, 9, 11, 14, 29, 37] and the references cited therein. We would like to point out that elegant analysis on residual-based a posteriori estimates of finite volume approximations for elliptic equations can be found in [1, 13].

Numerical solution of PDEs defined on smooth surfaces (or manifolds) in $\mathbb{R}^3$ recently attracted a lot of attentions due to its applications in various area, such as surface diffusion, global and local geophysical flows, ice formation, brain warping and so on [27, 32, 33, 34]. Thus it is of interest to investigate useful a posteriori error estimates for these type of problems. Some of *a priori* error analysis of second-order and fourth-order problems have been done for finite element methods [22, 23, 24] and finite volume methods [8, 20, 19]. Recently, a posteriori error estimates of finite element methods for discretizing the Laplace-Beltrami operator on surfaces were rigorously

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analyzed in [3, 15, 28] while similar study for finite volume methods is still lacking as far as we know. In this paper, we will rigorously derive a residual-based explicit a posteriori error estimator (in the sense of energy norm) for the finite volume discretization of the elliptic equations defined on a smooth surface which is represented as the zero level set of a signed distance function d to the surface.

The paper is organized as follows: in Section 1.1, we will briefly review the model problem defined on surfaces and define some notations to facilitate our analysis. Then the finite volume discretization of the problem is given in Section 2. In Section 3, we will derive a residual-based a posteriori estimate for the discretization in terms of the approximate solution, and prove its reliability and efficiency. Some numerical experiments are included in Section 4, to verify the theoretical results. In addition, we also numerically demonstrate that the a posteriori error estimator is quite *robust*, i.e., the constants in the a posteriori estimates are almost uniform across all test problems. Finally, concluding remarks are given in Section 5.

1.1 Model problem

Let $\mathbf{S}$ in $\mathbb{R}^3$ be an open bounded $C^{k,\alpha}$-hypersurface [22, 26] with $k \in \mathcal{N} \cup \{0\}$ and $0 \le \alpha < 1$, and we assume that $\mathbf{S}$ is represented globally by some *oriented distance function* (or say *level set function*) $d = d(\mathbf{x})$ defined in some open subset $\mathbf{U}$ of $\mathbb{R}^3$ such that $\mathbf{S} = \{\mathbf{x} \in \mathbf{U} | d(\mathbf{x}) = 0\}$ with $d \in C^{k,\alpha}$ and $\nabla d \neq 0$. The unit outward normal to $\mathbf{S}$ (with increasing d) at $\mathbf{x}$ is given by

$$\vec{\mathbf{n}}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x}), n_3(\mathbf{x})) = \frac{\nabla d(\mathbf{x})}{|\nabla d(\mathbf{x})|}$$

where $|\cdot|$ denotes the Euclidean norm and ∇ denotes the standard gradient operator in $\mathbb{R}^3$. Without loss of generality, we assume that $|\nabla d| \equiv 1$.

Let $\nabla_s = (\nabla_{s,1}, \nabla_{s,2}, \nabla_{s,3}) = \nabla - (\vec{\mathbf{n}} \cdot \nabla)\vec{\mathbf{n}}$ denote the tangential (surface) gradient operator, and $\Delta_s = \nabla_s \cdot \nabla_s$ be the so-called Laplace–Beltrami operator on $\mathbf{S}$ [26]. We use the standard notation for Sobolev spaces $L^p(\mathbf{S})$, $W^{m,p}(\mathbf{S})$, and $H^m(\mathbf{S}) = W^{m,2}(\mathbf{S})$ on $\mathbf{S}$. To make space $H^m(\mathbf{S})$ well defined, it is customary to assume $k + \alpha \ge max\{1, m\}$, see [31]. To avoid technical complexities, we further assume that $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth (say, of class C^3) and $\partial\mathbf{S} \neq \emptyset$ for the rest of the paper unless stated otherwise.

We consider the following steady diffusion-convection-reaction equation imposed on $\mathbf{S}$,

$$-\nabla_s \cdot (a(\mathbf{x})\nabla_s u(\mathbf{x})) + \nabla_s \cdot (\vec{v}(\mathbf{x})u(\mathbf{x})) + b(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \mathbf{S} \tag{1.1}$$

where the data in (1.1) is assumed to satisfy:

Assumption 1 *$f \in L^2(\mathbf{S})$, $a(\mathbf{x})$ is uniformly continuous on $\mathbf{S}$, $\vec{v} \in (W^{1,\infty}(\mathbf{S}))^3$, and $b \in L^\infty(\mathbf{S})$. Additionally, $a(\mathbf{x}) \ge \alpha_1 > 0$, $b(\mathbf{x}) \ge \alpha_2 \ge 0$ and $\nabla_s \cdot \vec{v}(\mathbf{x})/2 + b(\mathbf{x}) \ge \alpha_3 > 0$ for any $\mathbf{x} \in \mathbf{S}$.*

For simplicity, we take the homogeneous Dirichlet boundary condition:

$$u(\mathbf{x}) = 0, \qquad \forall\, \mathbf{x} \in \partial\mathbf{S}. \tag{1.2}$$

Note that our discussion here can be extended to more general cases such as $a = a(\mathbf{x})$ being a symmetric positive-definite tensor.

For any $u, \phi \in H_0^1(\mathbf{S})$, define the bilinear functional $\mathcal{A}$ to be

$$\mathcal{A}(u, \phi) = \int_{\mathbf{S}} a\nabla_s u \cdot \nabla_s \phi \, ds - \int_{\mathbf{S}} u\vec{v} \cdot \nabla_s \phi \, ds + \int_{\mathbf{S}} bu\phi \, ds, \tag{1.3}$$

then we have (for some generic constants $c_1 > 0$ and $c_2 > 0$)

$$\mathcal{A}(u,\phi) \leq c_1 \|u\|_{H^1(\mathbf{S})} \|\phi\|_{H^1(\mathbf{S})}, \tag{1.4}$$
$$\mathcal{A}(u,u) \geq c_2 \|u\|^2_{H^1(\mathbf{S})}. \tag{1.5}$$

Especially, let us define the *energy norm* $\|\cdot\|_E$ of u on $\mathbf{S}$ to be $\|u\|_E = (\mathcal{A}(u,u))^{1/2}$. Clearly the energy norm is equivalent to the H^1 norm under the given assumption.

We say that $u \in H^1_0(\mathbf{S})$ is a weak solution of the equation (1.1) if and only if

$$\mathcal{A}(u,\phi) = (f,\phi)_s, \qquad \forall \phi \in H^1_0(\mathbf{S}) \tag{1.6}$$

where

$$(f,\phi)_s = \int_{\mathbf{S}} f(\mathbf{x})\phi(\mathbf{x})\, ds.$$

The existence of the weak solution of equation (1.1) under Assumption 1 follows from the standard elliptic equation theory [26, 22].

Theorem 1 *Under Assumption 1, there exists a unique weak solution $u \in H^1_0(\mathbf{S})$ of (1.1). Moreover, $u \in H^2(\mathbf{S})$ and satisfies that*

$$\|u\|_{H^2(\mathbf{S})} \leq c\|f\|_{L^2(\mathbf{S})}\ . \tag{1.7}$$

for some generic constant $c > 0$.

We note that in the case of $\partial\mathbf{S} = \emptyset$, one can also show that, if $\alpha_2 > 0$ in Assumption 1, there exists a unique weak solution $u \in H^1(\mathbf{S})$ of (1.1).

2 Finite volume discretization

2.1 Piecewise linear approximation of the surface

We assume that $\mathbf{S}$ is a connected compact smooth hypersurface which is the zero level set of a signed distance function $|d(\mathbf{x})| = dist(\mathbf{x},\mathbf{S})$ defined on a strip (band)

$$\mathbf{U} = \{\mathbf{x} \in \Omega \quad | \quad dist(\mathbf{x},\mathbf{S}) < \delta\}, \qquad \text{for some } \delta > 0$$

around $\mathbf{S}$ such that there is a unique decomposition for any $\mathbf{x} \in \mathbf{U}$,

$$\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{p}(\mathbf{x}) \in \mathbf{S}$, $d(\mathbf{x})$ is the signed distance to $\mathbf{S}$, and $\vec{\mathbf{n}}(\mathbf{x})$ denotes the unit outward normal of $\mathbf{S}$ at $\mathbf{p}(\mathbf{x})$. The parameter δ can be determined by the surface curvatures (see [24]) if $\mathbf{S}$ is sufficiently smooth.

Denote by $\mathcal{T} = \{T_i\}_{i=1}^m$ the curved triangulation of the surface $\mathbf{S}$. And let $\mathbf{S}$ be approximated by a sequence of continuous piecewise linear complex $\{\mathbf{S}^h \subset \mathbf{U}\}$, consisting of a sequence of regular triangulations $\{\mathcal{T}^h = \{T_i^h\}_{i=1}^m\}$ with the mesh size approaching to zero. In order to avoid global double covering, we further assume that for each point $\mathbf{y} \in \mathbf{S}$ there is at most one point $\mathbf{x} \in \mathbf{S}^h$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{y}$, as suggested in [24]. Each T^h contains vertices $\{\mathbf{x}_i\}_{i=1}^n$ on $\mathbf{S}$ (i.e., $\{\mathbf{x}_i\}_{i=1}^n \subset \mathbf{S}\cap\mathbf{S}^h$), see Fig. 1(left). Clearly, $\mathbf{S}^h$ is globally of class $C^{0,1}$. We use $m(\cdot)$ to denote the area for planar

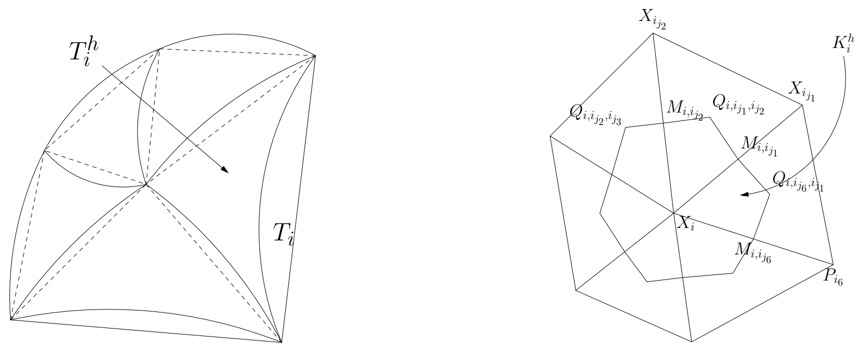

Figure 1: Approximate mesh surface and the control volume.

regions or the length for arcs and segments. Let h_i denote the size of a triangle $T_i^h \in \mathcal{T}^h$ and define $h = \max h_i$ to be the mesh size for $\mathcal{T}^h$. We say that $\mathcal{T}^h$ is shape-regular if for any $T_i^h \in \mathcal{T}^h$,

$$c_1 h \leq m(T_i^h) \leq c_2 h^2 \tag{2.2}$$

where c_1 and c_2 are positive constants independent of h. By the uniqueness of the decomposition discussed above, we define $T_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in T_i^h\}$, and let $\mathcal{T} = \{T_i\}_{i=1}^m$, then $\mathbf{S} = \cup_{i=1}^n T_i$. Note that this requires in particular that $\mathbf{p}(\partial \mathbf{S}^h) = \partial \mathbf{S}$.

Let the tangential gradient operator ∇_{s_h} on $\mathbf{S}^h$ be given by:

$$\nabla_{s_h} = (\nabla_{s_h,1}, \nabla_{s_h,2}, \nabla_{s_h,3}) = \nabla - \vec{\mathbf{n}}_h(\vec{\mathbf{n}}_h \cdot \nabla)$$

where $\vec{\mathbf{n}}_h(\mathbf{x}) = (n_{h1}(\mathbf{x}), n_{h2}(\mathbf{x}), n_{h3}(\mathbf{x}))$ is the unit outward normal to $\mathbf{S}^h$. Since $\vec{\mathbf{n}}_h$ is constant on each triangle T_i^h, ∇_{s_h} only needs to be locally defined as a two dimensional gradient operator on the plane formed by T_i^h, and the Sobolev space $W^{m,p}(\mathbf{S}^h)$ is well-defined for $m \leq 1$.

Denote by $\mathcal{U}$ the space of continuous piecewise linear polynomials on $\mathbf{S}^h$ with respect to $\mathcal{T}^h$, that is,

$$\mathcal{U} = \{U^h \in C^0(\mathbf{S}^h) \quad | \quad U^h|_{\partial \mathbf{S}^h} = 0, \quad U^h|_{T_i^h} \in \mathbb{P}_1(T_i^h)\} \tag{2.3}$$

where $\mathbb{P}_k(D)$ denote the space of polynomials of degree no larger than k on the planar domain D.

It is easy to see that $\mathcal{U} \subset H^1(\mathbf{S}^h)$ and for $U^h \in \mathcal{U}$ we have that $\nabla_{s_h} U^h$ is constant on each triangle $T_i^h \in \mathcal{T}^h$. A dual tessellation of $\mathcal{T}^h$ on $\mathbf{S}^h$ can be defined as shown in Fig. 1 (right). For each interior vertex $\mathbf{x}_i$, let $\chi_i = \{i_s\}_{s=1}^{m_i}$ be the set of indices of its neighbors, $Q_{i,i_j,i_{j+1}}$ (where $i_{s+1} = i_1$ if $s = m_i$) be the centroid of the triangle $\triangle \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}}$ and M_{i,i_j} be the midpoint of $\overline{\mathbf{x}_i \mathbf{x}_{i_j}}$ for $i_j \in \chi_i$. Let $K_i^h = \cup_{i_j \in \chi_i} \Omega_{i,i_j,i_{j+1}}$ where $\Omega_{i,i_j,i_{j+1}}$ denotes the polygonal region bounded by $\mathbf{x}_i$, M_{i,i_j}, $Q_{i,i_j,i_{j+1}}$ and $M_{i,i_{j+1}}$. In general, K_i^h is only piecewise planar and we define its projection onto $\mathbf{S}$ by $K_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in K_i^h\}$. Let σ denote the set of indices of all interior vertices of $\mathcal{T}^h$, then, $\mathcal{K} = \{K_i\}_{i\in\sigma}$ and $\mathcal{K}^h = \{K_i^h\}_{i\in\sigma}$ may be viewed as dual tessellations of $\mathbf{S} = \cup_{i=1}^m T_i$ and $\mathbf{S}^h = \cup_{i=1}^m T_i^h$. In the remaining part of this paper, for simplicity, we denote $i_{(j-1)\mathrm{mod}(m_i)+1}$ by i_j, if $j > m_i$ and $i_j \in \chi_i$ ($\mathbf{x}_{i_j}$ is a neighbor vertex of $\mathbf{x}_i$), otherwise denote $i_{(j-1)\mathrm{mod}(3)+1}$ by i_j, if $j > 3$ and $\mathbf{x}_{i_j}$ is a vertex of $T_i^h = \triangle \mathbf{x}_{i_1} \mathbf{x}_{i_2} \mathbf{x}_{i_3}$.

Denote by $\mathcal{V}$ the space of grid functions on $\mathbf{S}^h$ with respect to $\mathcal{K}^h$:

$$\mathcal{V} = \{V^h \quad | \quad V^h|_{K_i^h} \in \mathbb{P}_0(K_i^h)\}.$$

A set of basis functions $\{\Psi_i^h\}_{i\in\sigma}$ of $\mathcal{V}$ is given by

$$\Psi_i^h(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in K_i^h; \\ 0, & \mathbf{x} \in \mathbf{S}^h - K_i^h. \end{cases}$$

2.2 Generalized central scheme

We may uniquely extend a function ϕ defined on $\mathbf{S}$ to $\mathbf{U}$ by

$$\phi^l(\mathbf{x}) = \phi(\mathbf{p}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{U}. \tag{2.4}$$

Let $\mathbf{P} = \mathbf{I} - \vec{\mathbf{n}} \otimes \vec{\mathbf{n}}$ where $\otimes$ is the tensor or outer product defined as $\vec{a} \otimes \vec{b} = \vec{a}\vec{b}^T$, and it follows that

$$\nabla_s \phi = \nabla \phi^l - (\vec{\mathbf{n}} \cdot \nabla \phi^l)\vec{\mathbf{n}} = \mathbf{P}\nabla\phi^l, \tag{2.5}$$

due to (2.4).

We then do the similar extension from $\mathbf{S}^h$ to $\mathbf{U}$. Given a function ϕ_h defined on $\mathbf{S}^h$, first project it onto $\mathbf{S}$ by $\phi_h(\mathbf{y}) = \tilde{\phi}_h(\mathbf{p}(\mathbf{y}))$ for $\mathbf{y} \in \mathbf{S}^h$, then we apply (2.4) again to extend $\tilde{\phi}_h$ to $\mathbf{U}$, i.e.,

$$\phi_h^l(\mathbf{x}) = \tilde{\phi}_h(\mathbf{p}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{U}. \tag{2.6}$$

Then we successfully extend ϕ_h defined on $\mathbf{S}^h$ to $\mathbf{U}$ in two steps. Since all extensions of functions to $\mathbf{U}$ are constant along normals to $\mathbf{S}$, extensions of functions defined on $\mathbf{S}$ and of functions defined on $\mathbf{S}^h$ have the same properties.

With the above preparations, a generalized central finite volume scheme for the above steady diffusion-convection-reaction equation (1.1) can be defined as follows: find $u_h \in \mathcal{U}$ such that

$$\mathcal{A}_G^h(u_h, \phi_h) = (f^l, \phi_h)_{s_h} \qquad \forall\, \phi_h \in \mathcal{V}, \tag{2.7}$$

where

$$\mathcal{A}_G^h(u_h, \phi_h) = \sum_{i\in\sigma} \phi_{h,i}\mathcal{A}_G^h(u_h, \Psi_i^h)$$

and

$$\begin{aligned} \phi_{h,i} &= \phi_h(\mathbf{x}_i), \\ \mathcal{A}_G^h(u_h, \Psi_i^h) &= \int_{\partial K_i^h} (-a^l \nabla_{s_h} u_h + u_h \vec{v}^l) \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \int_{K_i^h} b^l u_h\, ds_h. \end{aligned}$$

The corresponding lifting u_h^l constrained on $\mathbf{S}$ can then be regarded as the approximate solution of the model problem (1.1). For the existence of the approximate solution u_h of (2.7) and related priori error estimates, see [19] for details. Specially, for the convection-dominated case, in order to eliminate non-physical oscillations, a up-wind finite volume scheme was given in [19]. In this paper, we will focus our discussion on the generalized central scheme.

3 A posteriori error estimators

Before deriving local a posteriori error estimates for the finite volume discretization (2.7), let us present some properties of lifts and extensions of functions defined above.

For $\mathbf{x} \in \mathbf{S}^h$, define

$$\mathbf{P}_h(\mathbf{x}) = \mathbf{I} - \vec{\mathbf{n}}_h(\mathbf{x}) \otimes \vec{\mathbf{n}}_h(\mathbf{x}), \tag{3.1}$$

and then for a function ϕ defined on $\mathbf{U}$, we have

$$\nabla_{s_h}\phi(\mathbf{x}) = \mathbf{P}_h \nabla\phi(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbf{S}^h. \tag{3.2}$$

According to (2.1) and (2.6), for ϕ_h defined on $\mathbf{S}^h$, it holds that

$$\nabla \phi_h^l(\mathbf{x}) = (\mathbf{P} - d\mathbf{H})\nabla\phi_h^l(\mathbf{p}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbf{S}^h, \tag{3.3}$$

where $\mathbf{H}:\mathbb{R}^3\to\mathbb{R}^3$ denotes the Weingarten map. Detailed discussions about $\mathbf{H}$ can be found in [15]. Since $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}}\equiv 1$, it holds $\vec{\mathbf{n}}\mathbf{H}=\mathbf{H}\vec{\mathbf{n}}=0$ and $\mathbf{PH}=\mathbf{HP}=\mathbf{H}$ (see [15]).

Then, for any $\mathbf{x}\in\mathbf{S}^h$, we obtain that

$$\nabla\phi_h^l(\mathbf{x})=(\mathbf{I}-d\mathbf{H})\mathbf{P}\nabla\phi_h^l(\mathbf{p}(\mathbf{x}))=(\mathbf{I}-d\mathbf{H})\nabla_s\phi_h^l(\mathbf{p}(\mathbf{x})). \tag{3.4}$$

Combination of (3.2) and (3.4) gives us

$$\nabla_{s_h}\phi_h(\mathbf{x})=\nabla_{s_h}\phi_h^l(\mathbf{x})=\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\nabla_s\phi_h^l(\mathbf{p}(\mathbf{x})). \tag{3.5}$$

Correspondingly, for $\psi\in H^1(\mathbf{S})$, we get

$$\nabla_{s_h}\psi^l(\mathbf{x})=\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\nabla_s\psi(\mathbf{p}(\mathbf{x})),\quad\forall\,\mathbf{x}\in\mathbf{U}.$$

If $\mathbf{x}\in\mathbf{S}^h$, (3.4) yields

$$\nabla_s\phi_h^l(\mathbf{p}(\mathbf{x}))=(\mathbf{I}-d\mathbf{H})^{-1}\nabla\phi_h^l(\mathbf{x}), \tag{3.6}$$

We note that the invertibility of $\mathbf{I}-d\mathbf{H}$ was proved in [15]. Next, we aim to deduce $\nabla_s\phi_h^l$ for given $\phi_h\in\mathbf{S}^h$. Using (3.5) and (3.6), we have

$$\nabla_{s_h}\phi_h(\mathbf{x})=\mathbf{P}_h\nabla\phi_h^l(\mathbf{x}),\quad\forall\,\mathbf{x}\in\mathbf{S}^h. \tag{3.7}$$

It is easy to see that for $\mathbf{x}\in\mathbf{S}^h$

$$0=\nabla\phi_h^l(\mathbf{x})\cdot\vec{\mathbf{n}}=\nabla_{s_h}\phi_h(\mathbf{x})\cdot\vec{\mathbf{n}}+(\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}})\nabla\phi_h^l(\mathbf{x})\cdot\vec{\mathbf{n}}_h,$$

then it follows that

$$\nabla\phi_h^l(\mathbf{x})\cdot\vec{\mathbf{n}}_h=-\frac{\nabla_{s_h}\phi_h(\mathbf{x})\cdot\vec{\mathbf{n}}}{\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}}}. \tag{3.8}$$

Thus we have

$$\nabla\phi_h^l(\mathbf{x})=\Big(\mathbf{I}-\frac{\vec{\mathbf{n}}_h\otimes\vec{\mathbf{n}}}{\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}}}\Big)\nabla_{s_h}\phi_h(\mathbf{x})$$

and

$$\nabla_s\phi_h^l(\mathbf{p}(\mathbf{x}))=(\mathbf{I}-d\mathbf{H})^{-1}\Big(\mathbf{I}-\frac{\vec{\mathbf{n}}_h\otimes\vec{\mathbf{n}}}{\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}}}\Big)\nabla_{s_h}\phi_h(\mathbf{x}).$$

Define

$$\mu_h(\mathbf{x})=\frac{ds(\mathbf{x})}{ds_h(p(\mathbf{x}))},\qquad\xi_h(\mathbf{x})=\frac{d\gamma(\mathbf{x}))}{d\gamma_h(p(\mathbf{x}))} \tag{3.9}$$

for any $\mathbf{x}\in\mathbf{S}^h$. Since $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth, we have that (see [19, 22])

$$|1-\mu_h(\mathbf{x})|\le ch^2,\qquad|1-\xi_h(\mathbf{x})|\le ch^2.$$

Finally, we cite the following results from [22] for later use:

Lemma 1 *For any $\phi\in H^1(\mathbf{S})$, there exist some generic constants $c_1,c_2,c_3,c_4>0$ such that*

$$\begin{cases} c_1\|\phi^l\|_{L^2(T_i^h)}\le\|\phi\|_{L^2(T_i)}\le c_2\|\phi^l\|_{L^2(T_i^h)},\\ c_3\|\phi^l\|_{H^1(T_i^h)}\le\|\phi\|_{H^1(T_i)}\le c_4\|\phi^l\|_{H^1(T_i^h)}\end{cases}$$

for any $T_i\in\mathcal{T}$.

With all the above notations, we have

$$\int_{\mathbf{S}^h} \nabla_{s_h}\phi_h \cdot \nabla_{s_h}\psi_h \, ds_h = \int_{\mathbf{S}} \mathbf{A}_h^l \nabla_s \phi_h^l \cdot \nabla_s \psi_h^l \, ds,$$

where

$$\mathbf{A}_h^l(\mathbf{p}(\mathbf{x})) = \mathbf{A}_h(\mathbf{x}) = \Big(\frac{1}{\mu_h}\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\Big)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbf{S}^h.$$

Similarly, it holds that

$$\int_{\mathbf{S}^h} \nabla_{s_h}\phi_h \psi_h \, ds_h = \int_{\mathbf{S}} \mathbf{B}_h^l \nabla_s \phi_h^l \psi_h^l \, ds,$$

where

$$\mathbf{B}_h^l(\mathbf{p}(\mathbf{x})) = \mathbf{B}_h(\mathbf{x}) = \Big(\frac{1}{\mu_h}\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\Big)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \mathbf{S}^h.$$

3.1 An a posteriori estimator and its reliability

In the following, we will derive a energy-type (or H^1-type) a posteriori estimate for the the discrete solution u_h, i.e., to estimate $\|u-u_h^l\|_E$ (or $\|u-u_h^l\|_{H^1(\mathbf{S})}$).

Let e be an edge shared by elements T_1 and T_2 on which have normals $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$, respectively, then we can define

$$[[a^l\nabla_{s_h}u_h - \vec{v}u_h]] = (a^l\nabla_{s_h}u_h - \vec{v}u_h)|_{T_1}\cdot\vec{\mathbf{n}}_1 + (a^l\nabla_{s_h}u_h - \vec{v}u_h)|_{T_2}\cdot\vec{\mathbf{n}}_2,$$

in particular, if $e \subset \partial\mathbf{S}_h$ we set $[[a^l\nabla_{s_h}u_h - \vec{v}u_h]] \equiv 0$. By Green's formula, it follows that for any $\phi \in H^1(\mathbf{S})$:

$$\begin{aligned}
\mathcal{A}(u-u_h^l,\phi) &= \int_{\mathbf{S}} f\phi\, ds - \mathcal{A}(u_h^l,\phi)\\
&= \int_{\mathbf{S}^h} f^l\mu_h\phi^l\, ds_h - \int_{\mathbf{S}} a\nabla_s u_h^l\cdot\nabla_s\phi\, ds + \int_{\mathbf{S}}(\vec{v}\cdot\nabla_s\phi)u_h^l\, ds - \int_{\mathbf{S}} bu_h^l\phi\, ds\\
&= \int_{\mathbf{S}^h} f^l\mu_h\phi^l\, ds_h - \int_{\mathbf{S}} a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l\cdot\nabla_s\phi\, ds + \int_{\mathbf{S}}(\mathbf{I}-\mathbf{B}_h^l)(\vec{v}\cdot\nabla_s\phi)u_h^l\, ds\\
&\quad - \int_{\mathbf{S}^h} a^l\nabla_{s_h}u_h\cdot\nabla_{s_h}\phi^l\, ds_h + \int_{\mathbf{S}^h}(\vec{v}^l\cdot\nabla_{s_h}\phi^l)u_h\, ds_h - \int_{\mathbf{S}} bu_h^l\phi\, ds\\
&= \int_{\mathbf{S}^h} f^l\mu_h\phi^l\, ds_h - \int_{\mathbf{S}} a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l\cdot\nabla_s\phi\, ds + \int_{\mathbf{S}}(\mathbf{I}-\mathbf{B}_h^l)(\vec{v}\cdot\nabla_s\phi)u_h^l\, ds\\
&\quad + \int_{\mathbf{S}^h}\nabla_{s_h}(a^l\nabla_{s_h}u_h)\phi^l\, ds_h - \sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h} a^l(\nabla_{s_h}u_h\cdot\vec{\mathbf{n}}_{T_i^h})\phi^l\, d\gamma_h\\
&\quad - \int_{\mathbf{S}^h}\nabla_{s_h}(\vec{v}^l u_h)\cdot\phi^l\, ds_h + \sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h} u_h\vec{v}^l\cdot\vec{\mathbf{n}}_{T_i^h}\cdot\phi^l\, d\gamma_h - \int_{\mathbf{S}} bu_h^l\phi\, ds.\\
&= \int_{S_h} f^l\mu_h\phi^l\, ds_h - \int_{S} a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l\nabla_s\phi\, ds + \int_{S}(\mathbf{I}-\mathbf{B}_h^l)(\vec{v}\cdot\nabla_s\phi)u_h^l ds\\
&\quad + \int_{\mathbf{S}^h}\nabla_{s_h}(a^l\nabla_{s_h}u_h)\phi^l\, ds_h - \int_{\mathbf{S}^h}\nabla_{s_h}(\vec{v}^l u_h)\cdot\phi^l\, ds_h\\
&\quad - \frac{1}{2}\sum_{T_i^h\in\mathcal{T}}\int_{\partial T_i^h}[[a^l\nabla_{s_h}u_h - \vec{v}u_h]]\phi^l d\,\gamma_h - \int_{S} bu_h^l\phi\, ds.
\end{aligned} \tag{3.10}$$

On the other hand, for any $u \in C^0(\mathbf{S}^h)$, denote by $\Pi(u)$ the interpolation of u onto $\mathcal{V}$, i.e. $\Pi(u) \in \mathcal{V}$ and $\Pi(u)(\mathbf{x}_i) = u(\mathbf{x}_i)$ for all $i \in \sigma$. Then it holds that

$$\begin{aligned} &-\sum_{i\in\sigma} \phi_i \int_{\partial K_i^h} a^l \nabla_{s_h} u_h \cdot \vec{\mathbf{n}}_{K^h}\, d\gamma_h \\ &= \sum_{T_i^h \in \mathcal{T}^h} \Big(-\sum_{j=1}^{3} \int_{\partial K_{ij}^h \cap T_i^h} (a^l \nabla_{s_h} u_h \cdot \vec{\mathbf{n}}_{K_{ij}^h}) \Pi(\phi^l)\, d\gamma_h \Big) \\ &= \sum_{T_i^h \in \mathcal{T}^h} \Big(-\int_{T_i^h} \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h) \Pi(\phi^l)\, ds_h + \int_{\partial T_i^h} (a^l \nabla_{s_h} u_h \cdot \vec{\mathbf{n}}_{T_i^h}) \Pi(\phi^l)\, d\gamma_h \Big) \end{aligned} \tag{3.11}$$

and similarly

$$\begin{aligned} &\sum_{i\in\sigma} \phi_i \int_{\partial K_i^h} u_h \vec{v}^l \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h \\ &\quad = \sum_{T_i^h \in \mathcal{T}^h} \Big(-\int_{T_i^h} \nabla_{s_h} \cdot (u_h \vec{v}^l) \Pi(\phi^l)\, ds_h + \int_{\partial T_i^h} (u_h \vec{v}^l \cdot \vec{\mathbf{n}}_{T_i^h}) \Pi(\phi^l)\, d\gamma_h \Big). \end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned} \mathcal{A}_G^h(u_h, \Pi(\phi^l)) &= -\sum_{i\in\sigma} \phi_i \int_{\partial K_i^h} a^l \nabla_{s_h} u_h \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \sum_{i\in\sigma} \phi_i \int_{\partial K_i^h} u_h \vec{v}^l \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \int_{\mathbf{S}_h} b^l u_h \Pi(\phi^l)\, ds_h \\ &= -\int_{\mathbf{S}_h} \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h) \Pi(\phi^l) ds_h + \int_{\mathbf{S}_h} \nabla_{s_h} \cdot (\vec{v}^l u_h) \Pi(\phi^l) ds_h + \int_{\mathbf{S}_h} b^l u_h \Pi(\phi^l)\, ds_h \\ &\quad + \frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} [[a^l \nabla_{s_h} u_h - \vec{v}^l u_h]] \phi^l\, d\gamma_h. \end{aligned} \tag{3.13}$$

Applying the equalities (3.10)–(3.13), we obtain

$$\begin{aligned} \mathcal{A}(u - u_h^l, \phi) &= \mathcal{A}(u - u_h^l, \phi) + \mathcal{A}_G^h(u_h, \Pi(\phi^l)) - \mathcal{A}_G^h(u_h, \Pi(\phi^l)) \\ &= \int_{\mathbf{S}^h} f^l \mu_h \phi^l\, ds_h - \int_{\mathbf{S}} a(\mathbf{P} - \mathbf{A}_h^l) \nabla_s u_h^l \cdot \nabla_s \phi\, ds + \int_{\mathbf{S}} (\mathbf{I} - \mathbf{B}_h^l)(\vec{v} \cdot \nabla_s \phi) u_h^l\, ds \\ &\quad + \int_{\mathbf{S}^h} \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h)(\phi^l - \Pi(\phi^l))\, ds_h - \int_{\mathbf{S}^h} \nabla_{s_h} \cdot (\vec{v}^l u_h)(\phi^l - \Pi(\phi^l))\, ds_h \\ &\quad - \frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} [[a^l \nabla_{s_h} u_h - u_h \vec{v}^l]](\phi^l - \Pi(\phi^l))\, d\gamma_h \\ &\quad - \int_{\mathbf{S}^h} b^l \mu_h u_h \phi^l\, ds_h + \int_{\mathbf{S}^h} b^l u_h \Pi(\phi^l) ds_h - \int_{\mathbf{S}^h} f^l \Pi(\phi^l)\, ds_h \\ &= \int_{\mathbf{S}^h} \big(f^l \mu_h + \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h - \vec{v} u_h) - b^l \mu_h u_h \big)(\phi^l - \Pi(\phi^l))\, ds_h \\ &\quad - \int_{\mathbf{S}} \big(a(\mathbf{P} - \mathbf{A}_h^l) \nabla_s u_h^l - (\mathbf{I} - \mathbf{B}_h^l) \vec{v} u_h^l \big) \cdot \nabla_s \phi\, ds \\ &\quad - \frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} [[a^l \nabla_s u_h - u_h \vec{v}^l]](\phi^l - \Pi(\phi^l)) d\, \gamma_h \\ &\quad + \int_{S_h} (1 - \mu_h)(b^l u_h - f^l) \Pi(\phi^l)\, ds_h \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{3.14}$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbf{S}^h} \big(f^l \mu_h + \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h - \vec{v} \cdot u_h) - b^l \mu_h u_h\big)(\phi^l - \Pi(\phi^l))\, ds_h, \\
I_2 &= -\frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} [[a^l \nabla_s u_h - u_h \vec{v}^{\,l}]](\phi^l - \Pi(\phi^l)) d\,\gamma_h, \\
I_3 &= -\int_{\mathbf{S}} \big(a(\mathbf{P} - \mathbf{A}_h^l)\nabla_s u_h^l - (\mathbf{I} - \mathbf{B}_h^l)\vec{v} u_h^l\big) \cdot \nabla_s \phi\, ds, \\
I_4 &= \int_{\mathbf{S}_h} (1 - \mu_h)(b^l u_h - f^l)\Pi(\phi^l)\, ds_h.
\end{aligned}
$$

Next, we will analyze the above four terms, respectively, to get an appropriate estimator. First, let us define

$$R = f^l \mu_h + \nabla_{s_h} \cdot (a^l \nabla_{s_h} u_h - \vec{v} u_h) - b^l \mu_h u_h \tag{3.15}$$

and

$$r = [[a^l \nabla_s u_h - u_h \vec{v}^{\,l}]]. \tag{3.16}$$

Then it follows that

$$
\begin{aligned}
|I_1| &= \Big| \sum_{T_i^h \in \mathcal{T}^h} \int_{T_i^h} R(\phi^l - \Pi(\phi^l))\, ds_h \Big| \\
&\leq \sum_{T_i^h \in \mathcal{T}^h} \|R\|_{L^2(T_i^h)} \|\phi^l - \Pi(\phi^l)\|_{L^2(T_i^h)} \\
&\leq c \sum_{T_i^h \in \mathcal{T}^h} h_i \|R\|_{L^2(T_i^h)} \|\phi\|_{H^1(T_i)},
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
|I_2| &= \Big| -\frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} r(\phi^l - \Pi(\phi^l))\, d\gamma_h \Big| \\
&\leq \frac{1}{2} \sum_{T_i^h \in \mathcal{T}^h} \|r\|_{L^2(\partial T_i^h)} \|\phi^l - \Pi(\phi^l)\|_{L^2(\partial T_i^h)} \\
&\leq c \sum_{T_i^h \in \mathcal{T}^h} \frac{1}{2} h_i^{\frac{1}{2}} \|r\|_{L^2(\partial T_i^h)} \|\phi\|_{H^1(T_i)}.
\end{aligned} \tag{3.18}
$$

Using Cauchy-Schwartz inequality and the trace theorem, we immediately get

$$
\begin{aligned}
|I_1 + I_2| &\leq c \sum_{T_i^h \in \mathcal{T}^h} (h_i \|R\|_{L^2(T_i^h)} + \frac{1}{2} h_i^{\frac{1}{2}} \|r\|_{L^2(\partial T_i^h)}) \|\phi\|_{H^1(T_i)} \\
&\leq c \Big(\sum_{T_i^h \in \mathcal{T}^h} h_i^2 \|R\|^2_{L^2(T_i^h)} + \frac{1}{4} \sum_{T_i^h \in \mathcal{T}^h} h_i \|r\|^2_{L^2(\partial T_i^h)} \Big)^{\frac{1}{2}} \|\phi\|_{H^1(\mathbf{S})}.
\end{aligned} \tag{3.19}
$$

As for I_3, we have

$$
\begin{aligned}
|I_3| &\le \sum_{T_i^h\in\mathcal{T}^h} \left\|a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l-(\mathbf{I}-\mathbf{B}_h^l)\vec{v}u_h^l\right\|_{L^2(T_i)}\|\nabla_s\phi\|_{L^2(T_i)}\\
&\le c\sum_{T_i^h\in\mathcal{T}^h} \Big\|a\sqrt{\mu_h}(\mathbf{P}-\mathbf{A}_h)(\mathbf{I}-d\mathbf{H})^{-1}\Big(\mathbf{I}-\frac{\vec{\mathbf{n}}_h\otimes\vec{\mathbf{n}}}{\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}}}\Big)\nabla_{s_h}u_h\\
&\qquad -\sqrt{\mu_h}[\mathbf{I}-\mathbf{B}_h]\vec{v}^l u_h\Big\|_{L^2(T_i^h)}\|\nabla_s\phi\|_{L^2(T_i)}. \qquad (3.20)
\end{aligned}
$$

Now set

$$
\mathbf{C}_h = a\sqrt{\mu_h}(\mathbf{P}-\mathbf{A}_h)(\mathbf{I}-d\mathbf{H})^{-1}\Big(\mathbf{I}-\frac{\vec{\mathbf{n}}_h\otimes\vec{\mathbf{n}}}{\vec{\mathbf{n}}_h\cdot\vec{\mathbf{n}}}\Big), \qquad (3.21)
$$

then we finally arrive at

$$
\begin{aligned}
|I_3| &\le c\sum_{T_i^h\in\mathcal{T}^h}\left\|\mathbf{C}_h\nabla_{s_h}u_h-\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}^l u_h\right\|_{L^2(T_i^h)}\|\phi^l\|_{H^1(T_i^h)}\\
&\le c\beta_1^*\left\|\mathbf{C}_h\nabla_{s_h}u_h-\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}^l u_h\right\|_{L^2(\mathbf{S}^h)}\|\phi\|_{H^1(\mathbf{S})} \qquad (3.22)
\end{aligned}
$$

where $\beta_1^*>0$ is a generic constant that also depends on the curvature information of the surface $\mathbf{S}$. For more discussions about this issue, see [15]. Also

$$
\begin{aligned}
|I_4| &= \Big|\int_{S_h}(1-\mu_h)(b^l u_h-f^l)\Pi(\phi^l)ds_h\Big|\\
&\le \sum_{T_i^h\in\mathcal{T}}\Big|\int_{T_i^h}(1-\mu_h)(b^l u_h-f^l)\Pi(\phi^l)ds_h\Big|\\
&\le \sum_{T_i^h\in\mathcal{T}}\|(1-\mu_h)(b^l u_h-f^l)\|_{L^2(T_i^h)}\|\Pi(\phi^l)\|_{L^2(T_i^h)}\\
&\le c\beta_2^*\Big(\sum_{T_i^h\in\mathcal{T}}\|(1-\mu_h)(b^l u_h-f^l)\|^2_{L^2(T_i^h)}\Big)^{\frac{1}{2}}\|\phi\|_{H^1(\mathbf{S})} \qquad (3.23)
\end{aligned}
$$

where $\beta_2^*>0$ is again a generic constant that also depends on the curvature information.

Thus, by letting $\phi=u-u_h^l$, we get an estimator as follows:

$$
\|u-u_h^l\|_E \le c\Big(\sum_{T_i^h\in\mathcal{T}^h}\mathbf{R}^2_{T_i^h,1}+\mathbf{R}^2_{T_i^h,2}+\mathbf{R}^2_{T_i^h,3}\Big)^{\frac{1}{2}} \qquad (3.24)
$$

where

$$
\begin{aligned}
\mathbf{R}_{T_i^h,1} &= \Big(h_i^2\|R\|^2_{L^2(T_i^h)}+\frac{1}{4}h_i\|r\|^2_{L^2(\partial T_i^h)}\Big)^{\frac{1}{2}},\\
\mathbf{R}_{T_i^h,2} &= \sqrt{\beta_1^*}\|\mathbf{C}_h\nabla_{s_h}u_h-\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{\mathbf{n}}^l\cdot u_h\|_{L^2(T_i^h)},\\
\mathbf{R}_{T_i^h,3} &= \sqrt{\beta_2^*}\|(1-\mu_h)(b^l u_h-f^l)\|_{L^2(T_i^h)}.
\end{aligned}
$$

Now let us formally define the local a posteriori error estimator $\eta_{T_i^h}$ on each triangle $T_i^h\in\mathcal{T}^h$ to be

$$
\eta^2_{T_i^h} = \mathbf{R}^2_{T_i^h,1}+\mathbf{R}^2_{T_i^h,2}+\mathbf{R}^2_{T_i^h,3} \qquad (3.25)
$$

and the following result is naturally obtained:

Theorem 2 *(Reliability of $\eta_{T_i^h}$) Assume that $u \in H_0^1(\mathbf{S})$ is the weak solution of the problem (1.6), and $u^h \in \mathcal{U}$ is the solution of the discrete problem (2.7). Then under Assumption 1, there exists a generic constant $c > 0$ such that*

$$\|u - u_h^l\|_E \leq c\eta_{\mathcal{T}^h} \tag{3.26}$$

where $\eta_{\mathcal{T}^h} = \big(\sum_{T_i^h \in \mathcal{T}^h} \eta_{T_i^h}^2\big)^{1/2}$.

Remark 1 *For any curved triangle $T_i \in \mathcal{T}$, one observes that*

$$|1 - \mu_h(\mathbf{x})| \leq ch_i^2, \qquad |d(\mathbf{x})| \leq ch_i^2, \qquad \|(\mathbf{P} - \mathbf{A}_h)(\mathbf{x})\|_{l^2 \to l^2} \leq ch_i^2 \tag{3.27}$$

for any $\mathbf{x} \in T_i$, then the following inequalities can be easily obtained

$$\|\mathbf{C}_h\|_{l^2 \to l^2} \leq c_1 \|\mathbf{P} - \mathbf{A}_h\|_{l^2 \to l^2} \leq c_2 h_i^2,$$

$$\|\mathbf{I} - \mathbf{B}_h\|_{l^2 \to l^2} \leq c_3 h_i^2.$$

Thus we know that the last two terms in (3.26) is of higher order compared with the first one.

3.2 Efficiency of the a posteriori estimator

In this section, we aim to prove the efficiency of the estimator derived in the previous section, i.e., $\eta_{T_i^h}$ does not overestimate the actual error.

Theorem 3 *Assume that $u \in H_0^1(\mathbf{S})$ is the solution of the problem (1.6), and $u^h \in \mathcal{U}$ is the solution of the discrete problem (2.7). We also assume that $\mathcal{T}$ is shape-regular. Then under Assumption 1, it holds that for any $T_i^h \in \mathcal{T}^h$,*

$$\begin{aligned} \eta_{T_i^h} \leq\ & c\Big[\|\mathbf{A}_h\|_{L^\infty(T^h)}^{1/2} \big(\|u - u_h^l\|_{H^1(T_i)} + \|\mathbf{C}_h \nabla_{s_h} u_h\|_{L^2(T_i^h)} + \|\sqrt{\mu_h}(\mathbf{I} - \mathbf{B})\vec{v}^l u_h\|_{L^2(T_i^h)}\big) \\ & + h_i \|R - \bar{R}\|_{L^2(T_i^h)} + h_i^{1/2} \|r - \bar{r}\|_{L^2(\partial T_i^h)}\Big] \end{aligned} \tag{3.28}$$

where c is a generic constant, and $\bar{R}$ and $\bar{r}$ are piecewise linear approximation of R and r, with respect to the triangulation $\mathcal{T}^h$.

Proof: The proof will follow the well-known frame work by Verfürth [36]. First, we aim to bound $\|R\|_{L^2(T_i^h)}$. Define the bubble function [36] $\phi_{T_i^h}$ on $T_i^h = \triangle x_{i_1} x_{i_2} x_{i_3}$ by $\phi_{T_i^h} = \prod_{j=1}^3 \zeta_{x_{i_j}}$, where each $\zeta_{x_{i_j}} \in \mathcal{U}$ and $\zeta_{x_i}(x_j) = \delta_{i,j}$, such that $\phi_{T_i^h}|_{\partial T_i^h} = 0$ and $\phi_{T_i^h} = 0$ outside T^h. Set $\bar{R} \in \mathcal{U}$ be a piecewise linear approximation to R on T_i^h. Let us take $\psi = \bar{R}\phi_{T_i^h}$ in (3.10), and apply Poincare inequality and Theorem 2.2 in [2], then we obtain

$$\begin{aligned} \int_{T_i^h} R\bar{R}\phi_{T_i^h}\, ds_h =\ & \int_{T_i} a\nabla_s(u - u_h^l) \cdot \nabla_s(\bar{R}^l \phi_{T_i^h}^l)\, ds - \int_{T_i} (u - u_h^l)\vec{v} \cdot \nabla_s(\bar{R}^l \phi_{T_i^h}^l)\, ds \\ & + \int_{T_i} b(u - u_h^l)\bar{R}^l \phi_{T_i^h}^l\, ds + \int_{T_i} a(\mathbf{P} - \mathbf{A}_h^l)\nabla_s u_h^l \cdot \nabla_s(\bar{R}^l \phi_{T_i^h}^l)\, ds \\ & - \int_{T_i} (\mathbf{I} - \mathbf{B}_h^l)\vec{v} u_h^l \cdot \nabla_s(\bar{R}^l \phi_{T_i^h}^l)\, ds \end{aligned}$$

and then

$$\begin{aligned}\Big|\int_{T_i^h} R\bar{R}\phi_{T_i^h}\,ds_h\Big| &\le \left(\|u-u_h^l\|_{H^1(T_i)} + \|a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l\|_{L^2(T_i)} + \|(\mathbf{I}-\mathbf{B}_h^l)\vec{v}u_h^l\|_{L^2(T_i)}\right)\\ &\quad \cdot\|\nabla_s(\bar{R}^l\phi_{T_i^h}^l)\|_{L^2(T_i)}\\ &\le \Big(\|u-u_h^l\|_{H^1(T_i)} + \|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(T_i^h)} + \|\sqrt{\mu_h}[\mathbf{I}-\mathbf{B}_h]\vec{v}^l u_h\|_{L^2(T_i^h)}\Big)\\ &\quad \cdot\|\mathbf{A}_h\|_{L^\infty(T^h)}^{1/2}\|\nabla_{s_h}(\bar{R}\phi_{T_i^h})\|_{L^2(T_i^h)}\\ &\le c\Big(\|u-u_h^l\|_{H^1(T_i)} + \|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(T_i^h)} + \|\sqrt{\mu_h}[\mathbf{I}-\mathbf{B}_h]\vec{v}^l u_h\|_{L^2(T_i^h)}\Big)\\ &\quad \cdot\|\mathbf{A}_h\|_{L^\infty(T_i^h)}^{1/2}h_i^{-1}\|\bar{R}\|_{L^2(T_i^h)} \qquad (3.29)\end{aligned}$$

where the constant c only depends on the shape regularity of $\mathcal{T}^h$.

Use Theorem 2.2 in [2] again, we then get

$$\begin{aligned}\|\bar{R}\|_{L^2(T_i^h)}^2 &\le c\|\bar{R}\sqrt{\phi_{T_i^h}}\|_{L^2(T_i^h)}^2\\ &\le c\Big(\int_{T_i^h} R\bar{R}\phi_{T_i^h}\,ds_h - \int_{T_i^h}\bar{R}(R-\bar{R})\phi_{T_i^h}\,ds_h\Big)\\ &\le c\Big(\int_{T_i^h} R\bar{R}\phi_{T_i^h}\,ds_h + \|R-\bar{R}\|_{L^2(T_i^h)}\|\bar{R}\phi_{T_i^h}\|_{L^2(T_i^h)}\Big)\\ &\le c\Big(\int_{T_i^h} R\bar{R}\phi_{T_i^h}\,ds_h + \|R-\bar{R}\|_{L^2(T_i^h)}\|\bar{R}\|_{L^2(T_i^h)}\Big). \qquad (3.30)\end{aligned}$$

Combination of (3.29) and (3.30) results in

$$\begin{aligned}\|\bar{R}\|_{L^2(T_i^h)}^2 \le c\Big(&\|R-\bar{R}\|_{L^2(T_i^h)} + h_{T_i^h}^{-1}\|\mathbf{A}_h\|_{L^\infty(T_i^h)}^{1/2}\big(\|u-u_h^l\|_{H^1(T_i)}\\ &+\|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(T_i^h)} + \|\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}u_h\|_{L^2(T_i^h)}\big)\Big)\|\bar{R}\|_{L^2(T_i^h)}. \qquad (3.31)\end{aligned}$$

Divided by $\|\bar{R}\|_{L^2(T_i^h)}$ and applying the triangle inequality, we immediately get

$$\begin{aligned}h_{T_i^h}\|R\|_{L^2(T_i^h)} &\le c\Big(h_{T_i^h}\|R-\bar{R}\|_{L^2(T_i^h)} + \|\mathbf{A}_h\|_{L^\infty(T_i^h)}^{1/2}\big(\|u-u_h^l\|_{H^1(T_i)}\\ &\quad +\|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(T_i^h)} + \|\sqrt{\mu_h}[\mathbf{I}-\mathbf{B}_h]\vec{v}u_h\|_{L^2(T_i^h)}\big)\Big). \qquad (3.32)\end{aligned}$$

Next, we try to bound the edge residual $\|r\|_{L^2(e)}$ where e is an edge shared by the triangle T_i^h and one of its neighbors, says T_j^h, and the closure of e contains the nodes x_i and x_j. Let $K^h = \bar{T}_i^h \cup \bar{T}_j^h$, and correspondingly $K = \bar{T}_i \cup \bar{T}_j$, denote ϕ_e the edge bubble function [36] over K^h defined as following: Take $\lambda_{i,1}, \lambda_{i,2}$ be the barycentric coordinate on T_i^h corresponding to x_i, define $\phi_e|_{T_i^h} = \lambda_{i,1}\lambda_{i,2}$; then we define $\phi_e|_{T_j^h}$ similarly. Thus $\phi_e|_{\partial K^h} = 0$, $\phi_e = 0$ outside K^h and $\phi_e > 0$ on e. Also let $\bar{r}$ be a piecewise linear approximation to r (i.e., $\bar{r} \in \mathcal{U}$), taking $\psi = \bar{r}\phi_e$ in (3.10), and applying Poincare inequality and Theorem 2.4 in [2], we obtain

$$\begin{aligned}\Big|\int_e r\bar{r}\phi_e\, d\gamma_h\Big| &= \Big| -\int_K a\nabla_s(u-u_h^l)\cdot\nabla_s(\bar{r}^l\phi_e^l)\, ds + \int_K (u-u_h^l)\vec{v}\cdot\nabla_s(\bar{r}^l\phi_e^l)\, ds \\ &\quad - \int_K b(u-u_h^l)\bar{r}^l\phi_e^l\, ds + \int_{K^h} R\bar{r}\phi_e\, ds_h \\ &\quad - \int_K a(\mathbf{P}-\mathbf{A}_h^l)\nabla_s u_h^l\nabla_s(\bar{r}^l\phi_e^l)\, ds \\ &\quad + \int_K (\mathbf{I}-\mathbf{B}_h^l)(\vec{v}\nabla_s(\bar{r}^l\phi_e^l))u_h^l\, ds\Big| \\ &\le \|R\|_{L^2(K^h)} h_i^{1/2}\|\bar{r}\|_{L^2(e)} + c\Big(\|u-u_h^l\|_{H^1(K)} + \|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(K^h)} \\ &\quad + \|\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}^l u_h\|_{L^2(K^h)}\Big)\|\mathbf{A}_h\|^{1/2}_{L^\infty(K^h)}\|\nabla_{s_h}(\bar{r}\phi_e)\|_{L^2(K^h)} \qquad (3.33)\end{aligned}$$

where the constant c only depends on the shape regularity of the mesh. Again, using Theorem 2.4 in [2], we have

$$\|\nabla_{s_h}(\bar{r}\phi_e)\|_{L^2(K^h)} \le c h_i^{-1/2}\|\bar{r}\|_{L^2(e)}, \qquad (3.34)$$

and

$$\|\bar{r}\|^2_{L^2(e)} \le c\int_e \bar{r}^2\phi_e d\gamma_h \le c\Big(\int_e r\bar{r}\phi_e d\gamma_h + \|r-\bar{r}\|_{L^2(e)} h_i^{1/2}\|\bar{r}\|_{L^2(e)}\Big). \qquad (3.35)$$

Combining (3.33)-(3.35), we then get

$$\begin{aligned} h^{1/2}_{T_i^h}\|\bar{r}\|^2_{L^2(e)} \le c\Big(&\|\mathbf{A}_h\|^{1/2}_{L^\infty(K^h)}\big(\|u-u_h^l\|_{H^1(K)} + \|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(K^h)} + \|\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}^l u_h\|_{L^2(K^h)}\big) \\ &+ h_{T^h}\|R\|_{L^2(K^h)} + h^{1/2}_{T^h}\|r-\bar{r}\|_{L^2(e)}\Big)\|\bar{r}\|_{L^2(e)}. \qquad (3.36)\end{aligned}$$

and consequently

$$\begin{aligned} h^{1/2}_{T_i^h}\|r\|_{L^2(e)} \le c\Big(&\|\mathbf{A}_h\|^{1/2}_{L^\infty(K^h)}\big(\|u-u_h^l\|_{H^1(K)} + \|\mathbf{C}_h\nabla_{s_h}u_h\|_{L^2(K^h)} + \|\sqrt{\mu_h}(\mathbf{I}-\mathbf{B}_h)\vec{v}^l u_h\|_{L^2(K^h)}\big) \\ &+ h_{T^h}\|R\|_{L^2(K^h)} + h^{1/2}_{T^h}\|r-\bar{r}\|_{L^2(e)}\Big). \qquad (3.37)\end{aligned}$$

Notice that $\mathbf{R}_{T_i^h,2}$ and $\mathbf{R}_{T_i^h,3}$ are higher-order terms compared with $\mathbf{R}_{T_i^h,1}$ from the discussion in Remark 1, and we obtain the efficiency relation (3.28) directly from (3.32) and (3.37). □

4 Numerical Experiments

In this section, several numerical experiments are presented to verify the reliability and efficiency of the a posteriori estimator proposed in the previous section. All experiments are performed for the model equation(1.1) with a given exact solution $u(\mathbf{x})$. Boundary conditions are set correspondingly if $\partial\mathbf{S} \neq \emptyset$. In each experiment, the initial mesh is generated by the so-called constrained centroidal Voronoi Delaunay triangulation (CCVDT) algorithm [16] with a uniform density function. The mesh refinement at each level is done by applying the marking strategy used in [15, 35] with the parameter $\theta = 0.3$ (θ is used to control the refinement process [21]). The refinement process stops after the number of nodes reaches 30,000 for each of the example. We set $\beta_1^* = \beta_2^* = 1$ in (3.25) for our numerical experiments. Recently, the study of *robustness* of a posteriori error estimates also attracted much attention, i.e., whether the constants in the a posteriori estimate are almost uniform for a class of similar problems. We will also numerically address this problem.

4.1 Example 1: Half-sphere

In this experiment, the surface $\mathbf{S}$ is taken to be the northern half of the unit sphere, i.e.,

$$\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 \geq 0\},$$

and its boundary is given by

$$\partial\mathbf{S} = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 0\}.$$

The outer normal at $\mathbf{x} \in \mathbf{S}$ is simply (x_1, x_2, x_3). We set $a(\mathbf{x}) = 1$, $\vec{v}(\mathbf{x}) = (1, 2, 3)$, $b(\mathbf{x}) = 1$, and the exact solution u is chosen to be

$$u(\mathbf{x}) = \frac{1}{x_1^2 + x_2^2 + (1 - x_3)^2 + 0.02}.$$

Clearly u has a peak at $(0, 0, 1)$, the top of the half sphere.

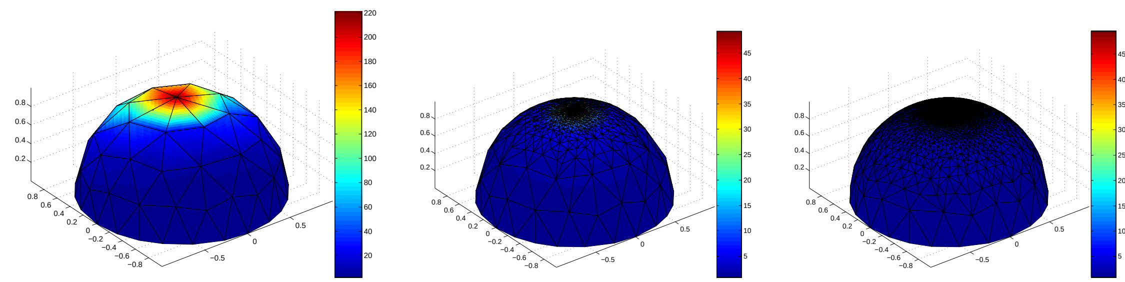

Figure 2: (From left to right) The initial mesh with 64 nodes and adaptively refined meshes with 817 nodes (Level 12) and 4098 nodes (Level 16) for Example 1.

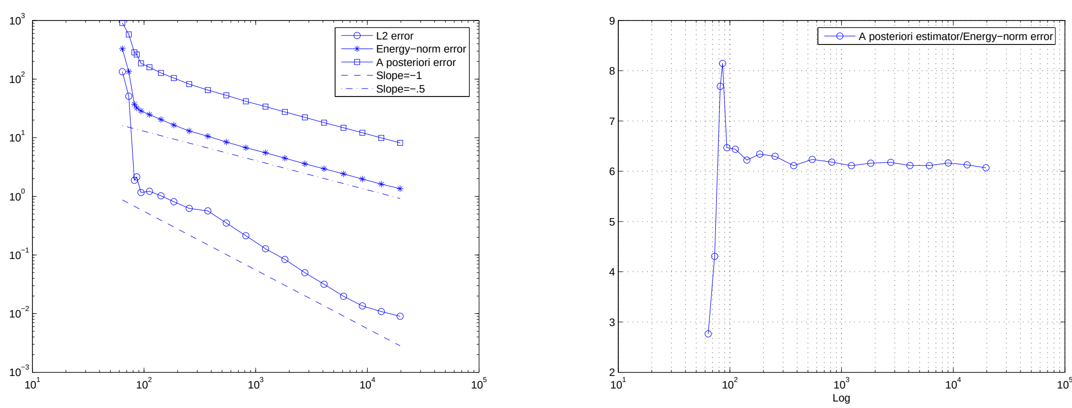

Figure 3: Comparison of L^2, energy-norm errors and the a posteriori error estimator η at all levels for Example 1.

The first experiment takes all the coefficients to be constant. The initial mesh and the refined meshes at steps 12 and 16 are shown in Fig. 2. It can be easily seen that the meshes around the peak of the exact solution are well refined. In Fig. 3(left), we draw the L^2 and energy-norm errors of the approximate solution u_h at all steps and also the a posteriori error estimate $\eta_{\mathcal{T}^h}$, along with some reference slopes. From Fig. 3 it can be concluded that the approximate solutions converge in the second-order measured under L^2 norm and the first-order under energy norm. It is also obvious that our error estimator has the same convergence rate as the energy-norm error, see Fig. 3(right) for details where the ratio between the error estimator η and the energy-norm error at each step of refinements are presented. From this figure, we can easily observe that the ratio is quickly stable down after the oscillations at the first few refinements and converges to a constant around 6.1.

4.2 Example 2: Cornered surface

The surface $\mathbf{S}$ is now selected to be

$$\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3 \mid (x_3 - x_2^2)^2 + x_1^2 + x_2^2 = 1,\ x_3 \geq x_2^2,\ x_1 \geq 0 \text{ or } x_2 \geq 0\}$$

with the boundary

$$\begin{aligned}\partial\mathbf{S} = &\{(x_1, x_2, x_2^2 + \sqrt{1 - x_1^2 + x_2^2}) \mid x_1^2 + x_2^2 = 1,\ x_1 \geq 0 \text{ or } x_2 \geq 0\} \\ &\cup\{(0, x_2, x_2^2 + \sqrt{1 + x_2^2}) \mid -1 \leq x_2 \leq 0\} \cup \{(x_1, 0, \sqrt{1 - x_1^2}) \mid -1 \leq x_1 \leq 0\}.\end{aligned}$$

Clearly, the boundary of $\mathbf{S}$ is not smooth at $(0, 0, 1)$. The outer normal at $\mathbf{x} \in \mathbf{S}$ is now given by $\vec{\mathbf{n}}(\mathbf{x}) = \vec{t}/\|\vec{t}\|$ with $\vec{t} = (x_1, x_2(1 - 2(x_3 - x_2^2)), x_3 - x_2^2)$. We use variant coefficients $a(\mathbf{x}) = 2 + x_1 x_2$, $\vec{v}(\mathbf{x}) = (1 + x_1, 2 + x_2, 3 + x_3)$ and $b(\mathbf{x}) = 1$. The exact solution u is set again to be

$$u(\mathbf{x}) = \frac{1}{x_1^2 + x_2^2 + (1 - x_3)^2 + 0.02}.$$

Then the peak of error occurs at the corner $(0, 0, 1)$.

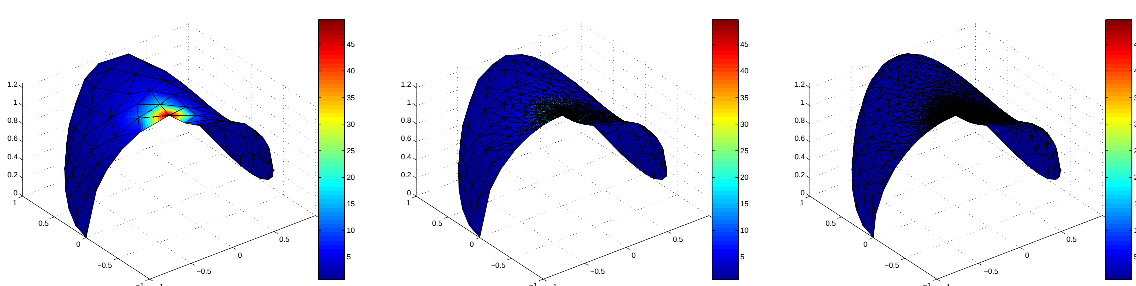

Figure 4: (From left to right) The initial mesh with 108 nodes and adaptively refined meshes with 929 nodes (Level 11) and 3067 nodes (Level 14) for Example 2.

In this experiment, we choose coefficients a and $\vec{v}$ to be dependent on $\mathbf{x}$ (non-constants), and the boundary is not globally smooth as in experiment 1, but with a corner. We can see that the a posteriori estimate is still very effective, as shown in Fig. 4, where the meshes around the peak of exact solution are refined much more heavily, as expected. In Fig. 5(left), we see that our estimator remains the same convergence rate as the energy-norm error, and the L^2 error maintains a second-order convergence. And as in experiment 1, the Fig. 5(right) gives us a steady almost-constant ratio between the estimator $\eta_{\mathcal{T}^h}$ and the energy-norm error, after oscillations of the first few steps of mesh refinement. Notice that the ratio is around 8.3 for this example which is quite close to that of Example 1.

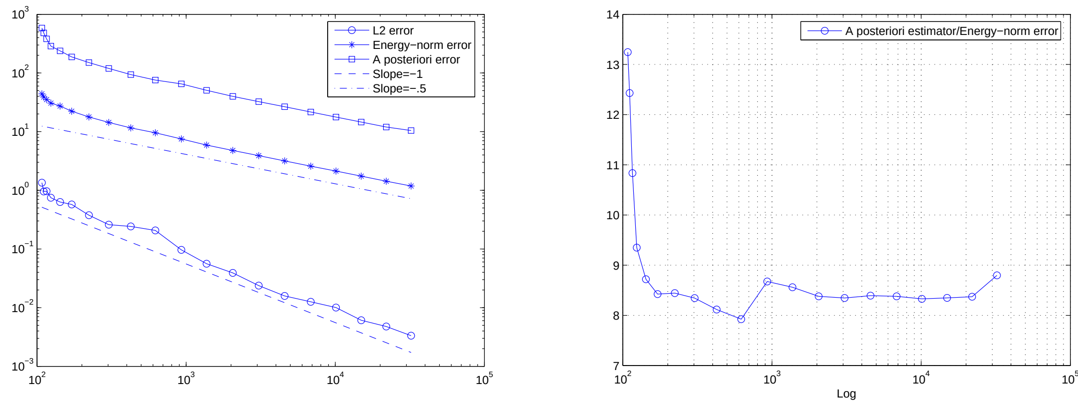

Figure 5: Comparison of L^2, Energy-norm errors and the a posteriori estimator η at all levels for Example 2.

4.3 Example 3: Torus

In the third experiment, the surface $\mathbf{S}$ is taken to be a torus such as

$$\mathbf{S} = \left\{ \mathbf{x} \in \mathbb{R}^3 \mid (x_1 - \frac{(r_1+r_2)x_1}{2\rho})^2 + (x_2 - \frac{(r_1+r_2)x_2}{2\rho})^2 + x_3^2 = \frac{(r_2-r_1)^2}{4} \right\}$$

where $\rho = \sqrt{x_1^2 + x_2^2}$, $r_1 = 0.5$, and $r_2 = 1$. Clearly, this surface has no boundary, and we will show that our estimator still works well.

The outer normal at $\mathbf{x} \in \mathbf{S}$ is given by $\vec{\mathbf{n}}(\mathbf{x}) = \vec{t}/|\vec{t}|$ with

$$\vec{t} = \begin{pmatrix} (x_1 - \tilde{x}_1)\left(1.0 - \frac{r_1+r_2}{2\rho} + \frac{(r_1+r_2)x_1^2}{2\rho^3}\right) + (x_2 - \tilde{x}_2)\left(\frac{(r_1+r_2)x_2x_1}{2\rho^3}\right) \\ (x_2 - \tilde{x}_2)\left(1.0 - \frac{r_1+r_2}{2\rho} + \frac{(r_1+r_2)x_2^2}{2\rho^3}\right) + (x_1 - \tilde{x}_1)\left(\frac{(r_1+r_2)x_2x_1}{2\rho^3}\right) \end{pmatrix},$$

where $\tilde{x}_1 = (r_1+r_2)x_1/2\rho$, $\tilde{x}_2 = (r_1+r_2)x_2/2\rho$. We let $a(\mathbf{x}) = 1 + x_1^2$, $\vec{v} = (0,0,0)$ and $b(\mathbf{x}) = 1 + x_1^2 + x_2^2 + x_3^2$, and the exact solution u is set to be

$$u(\mathbf{x}) = e^{\frac{1}{(x_1+1)^2 + x_2^2 + x_3^2 + 0.25}}.$$

Obviously u has a peak at $(-1,0,0)$.

Fig. 6 shows that the meshes around the peak are well-refined. Fig. 7 verifies the expected convergence rates of L^2, energy-norm errors and the a posteriori error estimate, as well as the almost-constant ratio between the error estimator $\eta_{\mathcal{T}^h}$ and the energy-norm error. More specifically, the ratio stays around 7.5 with small perturbations that is again quite close to that of Example 1 and Example 2. This numerical observation tells us that the proposed a posteriori error estimate is also quite robust in applications. In this experiment, we show that the proposed error estimator behaves very well even if we have a closed surface. Actually, we would like to note that the same theoretical results can be obtained for closed surfaces with similar analysis as we did in the previous sections.

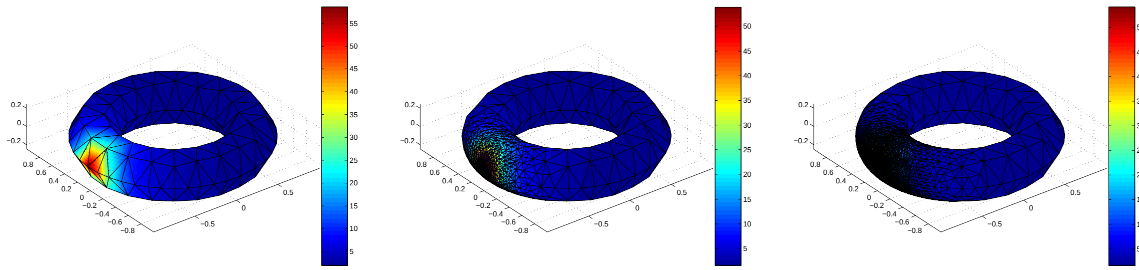

Figure 6: (From left to right) The initial mesh with 146 nodes and adaptively refined meshes with 869 nodes (Level 9) and 4190 nodes (Level 13) for Example 3.

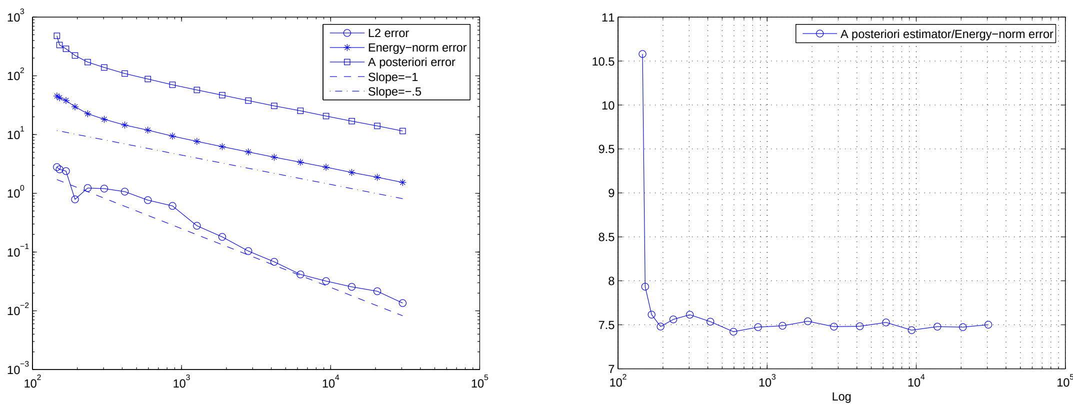

Figure 7: Comparison of L^2, energy-norm error and the a posteriori error estimator η at all levels for Example 3.

5 Conclusions

In this paper, we derive a residual-based explicit a posteriori error estimate for the finite volume discretization of elliptic partial differential equations defined on a smooth surface in $\mathbb{R}^3$. We rigorously prove both the reliability and the efficiency of the proposed error estimator and verified the theoretical results through numerical examples. The numerical results also demonstrate the robustness of the error estimator. The on-going and future works involve studying similar a posteriori error estimators on finite volume approximation for higher-order and time-dependent problems defined on surfaces.

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