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SOME INEQUALITIES FOR THE TENSOR PRODUCT OF GREEDY BASES AND WEIGHT-GREEDY BASES

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Abstract. In this paper we study properties of bases that are important in nonlinear m-term approximation with regard to these bases. It is known that the univariate Haar basis is a greedy basis for $L_p([0,1))$, $1 < p < \infty$. This means that a greedy type algorithm realizes nearly best m-term approximation for any individual function. It is also known that the multivariate Haar basis that is a tensor product of the univariate Haar bases is not a greedy basis. This means that in this case a greedy algorithm provides a m-term approximation that may be equal to the best m-term approximation multiplied by a growing (with m) to infinity factor. There are known results that describe the behavior of this extra factor for the Haar basis. In this paper we extend these results to the case of a basis that is a tensor product of the univariate greedy bases for $L_p([0,1))$, $1 < p < \infty$. Also, we discuss weight-greedy bases and prove a criterion for weight-greedy bases similar to the one for greedy bases.

1. Introduction

In this paper we study properties of bases that are important in nonlinear m-term approximation with regard to these bases. We begin with a brief historical survey that provides a motivation for our investigation. Also, this research is motivated by applications in nonparametric statistics. We plan to report the corresponding applications in statistics in our next paper. We remind the definition of the univariate Haar basis. Denote $\mathcal{H} := \{H_k\}_{k=1}^\infty$ the Haar basis on $[0,1)$ normalized in $L_2([0,1))$: $H_1 = 1$ on $[0,1)$ and for $k = 2^n + l$, $n = 0,1,\dots$, $l = 1,2,\dots,2^n$

$$H_k(x) = \begin{cases} 2^{n/2}, & x \in [(2l-2)2^{-n-1}, (2l-1)2^{-n-1}) \\ -2^{n/2}, & x \in [(2l-1)2^{-n-1}, 2l2^{-n-1}) \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{H}_p := \{H_{k,p}\}_{k=1}^\infty$ the Haar basis $\mathcal{H}$ renormalized in $L_p([0,1))$.

The following important property of the Haar basis (the Haar basis is a democratic basis) has been established in [T1]: for any Λ, $|\Lambda| = m$, one has

$$C_1(p)m^{1/p} \le \|\sum_{k\in\Lambda} H_{k,p}\|_p \le C_2(p)m^{1/p}, \quad 1 < p < \infty. \tag{1.1}$$

Our main interest in this paper is to study the multivariate bases. There are two standard ways to build a multivariate Haar basis. One way is based on the idea of multiresolution analysis. In this way we obtain a multivariate Haar basis consisting of functions whose supports are dyadic cubes. The theory of greedy approximation in this case is parallel to the univariate case (see [T1], [CDH]). In this paper we use the tensor product of the univariate bases as a way of building a multivariate basis.

We define the multivariate Haar basis $\mathcal{H}_p^d$ as the tensor product of the univariate Haar bases: $\mathcal{H}_p^d := \mathcal{H}_p \times \cdots \times \mathcal{H}_p$; $H_{\mathbf{n},p}(x) := H_{n_1,p}(x_1)\cdots H_{n_d,p}(x_d)$, $x = (x_1, \dots, x_d)$, $\mathbf{n} = (n_1, \dots, n_d)$. Supports of functions $H_{\mathbf{n},p}$ are arbitrary dyadic parallelepipeds (intervals). It is known (see [T3]) that the tensor product structure of the multivariate wavelet bases makes them universal for approximation of anisotropic smoothness classes with different anisotropy. It is also known that the study of such bases is more difficult than the study of the univariate bases. In many cases we need to develop new technique and in some cases we encounter with new phenomena. For instance, it turned out that the property (1.1) does not hold for the multivariate Haar basis $\mathcal{H}_p^d$ for $p \neq 2$ (see [T4] for a detailed discussion). It is known from [T2], [W], and [KaT] that the function

$$\mu(m, \mathcal{H}_p^d) := \sup_{k\le m}\Big(\sup_{\Lambda:|\Lambda|=k} \|\sum_{\mathbf{n}\in\Lambda} H_{\mathbf{n},p}\|_p / \inf_{\Lambda:|\Lambda|=k} \|\sum_{\mathbf{n}\in\Lambda} H_{\mathbf{n},p}\|_p\Big)$$

plays a very important role in estimates of the m-term greedy approximation in terms of the best m-term approximation. For instance (see [T2]),

$$\|f - G_m^{L_p}(f, \mathcal{H}_p^d)\|_p \le C(p,d)\mu(m, \mathcal{H}_p^d)\sigma_m(f, \mathcal{H}_p^d)_p, \quad 1 < p < \infty. \tag{1.2}$$

The greedy approximant $G_m^{L_p}(f, \mathcal{H}_p^d)$ and the best m-term approximation $\sigma_m(f, \mathcal{H}_p^d)_p$ are defined below. The following theorem gives, in particular, the upper estimates for the $\mu(m, \mathcal{H}_p^d)$.

Theorem A. *Let $1 < p < \infty$. Then for any Λ, $|\Lambda| = m$, we have for $2 \le p < \infty$*

$$C_{p,d}^1 m^{1/p} \min_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}| \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} H_{\mathbf{n},p}\|_p \le C_{p,d}^2 m^{1/p} (\log m)^{h(p,d)} \max_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}|,$$

and for $1 < p \le 2$

$$C_{p,d}^3 m^{1/p} (\log m)^{-h(p,d)} \min_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}| \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} H_{\mathbf{n},p}\|_p \le C_{p,d}^4 m^{1/p} \max_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}|$$

where $h(p,d) := (d-1)|1/2 - 1/p|$.

Theorem A for $d = 1$, $1 < p < \infty$ has been proved in [T1], in the case $d = 2$, $4/3 \le p \le 4$ it has been proved in [T2]. Theorem A in the general case has been proved in [W]. It is known ([T4]) that the extra log factors in Theorem A are sharp.

In Section 2 we generalize Theorem A to the case of basis that is the tensor product of greedy bases. We now give the corresponding definitions and introduce notations. We will do this in a general setting.

Let X be an infinite dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_n\}_{n=1}^{\infty}$ be a normalized basis for X ($\|\psi_n\| = 1$, $n \in \mathbb{N}$). For a given $f \in X$ we define the best m-term approximation with regard to Ψ as follows

$$\sigma_m(f,\Psi) := \sigma_m(f,\Psi)_X := \inf_{b_k,\Lambda} \|f - \sum_{k\in\Lambda} b_k\psi_k\|_X,$$

where inf is taken over coefficients b_k and sets of indices Λ with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f,\Psi)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \dots$, of the positive integers decreasing and write $\rho \in D(f)$ if

$$|c_{k_1}(f,\Psi)| \ge |c_{k_2}(f,\Psi)| \ge \dots \quad .$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the mth greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by the formula

$$G_m(f,\Psi) := G_m^X(f,\Psi) := G_m(f,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(f,\Psi)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme (it is not computationally ready) for m-term approximation of an element f. This algorithm is known in the theory of nonlinear approximation under the name Greedy Algorithm (see for instance [T1], [T2], [W]) and under the more specific name Thresholding Greedy Algorithm (TGA) (see [T4], [DKKT]). We will use the latter name in this paper. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f,\Psi,\rho)\|_X = \sigma_m(f,\Psi)_X,$$

or a little weaker

$$\|f - G_m(f,\Psi,\rho)\|_X \le G\sigma_m(f,\Psi)_X$$

for all elements $f \in X$ with a constant $G = C(X,\Psi)$ independent of f and m. The following concept of greedy basis has been introduced in [KT].

Definition 1. *We call a normalized basis Ψ greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$\|f - G_m(f,\Psi,\rho)\|_X \le G\sigma_m(f,\Psi)_X$$

holds with a constant independent of f, m.

The first result in this direction (see [T1]) established that the univariate Haar basis is a greedy basis.

Let Ψ be a normalized basis for $L_p([0,1))$. For the space $L_p([0,1)^d)$ we define $\Psi^d := \Psi \times \cdots \times \Psi$ (d times); $\psi_{\mathbf{n}}(x) := \psi_{n_1}(x_1)\cdots\psi_{n_d}(x_d)$, $x = (x_1,\dots,x_d)$, $\mathbf{n} = (n_1,\dots,n_d)$. In Section 2 we prove the following theorem using the scheme of the prove similar to that from [W].

Theorem 1. *Let $1 < p < \infty$ and let Ψ be a greedy basis for $L_p([0,1))$. Then for any Λ, $|\Lambda| = m$, we have for $2 \le p < \infty$*

$$C^5_{p,d} m^{1/p} \min_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}| \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} \psi_{\mathbf{n}}\|_p \le C^6_{p,d} m^{1/p} (\log m)^{h(p,d)} \max_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}|,$$

and for $1 < p \le 2$

$$C^7_{p,d} m^{1/p} (\log m)^{-h(p,d)} \min_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}| \le \|\sum_{\mathbf{n}\in\Lambda} c_{\mathbf{n}} \psi_{\mathbf{n}}\|_p \le C^8_{p,d} m^{1/p} \max_{\mathbf{n}\in\Lambda} |c_{\mathbf{n}}|$$

where $h(p,d) := (d-1)|1/2 - 1/p|$.

The inequality (1.2) has been extended in [W] to a normalized unconditional basis Ψ for X instead of $\mathcal{H}^d_p$ for $L_p([0,1)^d)$. Therefore, as a corollary of Theorem 1 we obtain the following inequality for a greedy basis Ψ (for $L_p([0,1))$)

$$\|f - G^{L_p}_m(f, \Psi^d)\|_p \le C(\Psi, d, p)(\log m)^{h(p,d)} \sigma_m(f, \Psi^d)_p, \quad 1 < p < \infty. \tag{1.3}$$

In Section 3 we prove a generalization of Theorem A to the case of $H_{\mathbf{n},q}$ instead of $H_{\mathbf{n},p}$. It will be convenient for us to enumerate the Haar system by the dyadic intervals. We set $h_{[0,1]} := H_{1,\infty}$; $h_{[(l-1)2^{-n}, l2^{-n})} := H_{2^n+l,\infty}$, $l = 1, \dots, 2^n$, $n = 0, 1, \dots$; $h_I(x) := h_{I_1}(x_1) \dots h_{I_d}(x_d)$, $I = I_1 \times \dots \times I_d$.

An interesting generalization of m-term approximation was considered in [CDH]. Let $\Psi = \{\psi_I\}_I$ be a basis indexed by dyadic intervals. Take an α and assign to each index set Λ the following measure

$$\Phi_\alpha(\Lambda) := \sum_{I\in\Lambda} |I|^\alpha.$$

In the case $\alpha = 0$ we get $\Phi_0(\Lambda) = |\Lambda|$. An analog of best m-term approximation is the following

$$\inf_{\Lambda:\Phi_\alpha(\Lambda)\le m} \inf_{c_I, I\in\Lambda} \|f - \sum_{I\in\Lambda} c_I \psi_I\|_p.$$

A detailed study of this type of approximation (restricted approximation) can be found in [CDH]. The following theorem proved in Section 3 provides the inequalities useful in the study of restricted approximation with regard to the $\mathcal{H}^d_p$.

Theorem 2. *Let $1 < p < \infty$. Then for any $a > 0$ and any Λ, $|\Lambda| = m$ we have for $2 \le p < \infty$*

$$\sum_{I\in\Lambda} \||I|^{-a} h_I\|_p^p \ll \|\sum_{I\in\Lambda} |I|^{-a} h_I\|_p^p \ll (\log m)^{(1/2-1/p)p(d-1)} \sum_{I\in\Lambda} \||I|^{-a} h_I\|_p^p, \tag{1.4}$$

and for $1 < p \le 2$

$$(\log m)^{(1/2-1/p)p(d-1)} \sum_{I\in\Lambda} \||I|^{-a} h_I\|_p^p \ll \|\sum_{I\in\Lambda} |I|^{-a} h_I\|_p^p \ll \sum_{I\in\Lambda} \||I|^{-a} h_I\|_p^p. \tag{1.5}$$

Here, the sign $\ll$ means that we have the corresponding inequality with an extra factor that does not depend on m and Λ. We note that Theorem 2 in the case $a = 1/p$ coincides with Theorem A. Theorem 2 in the case $d = 1$ has been proved in [CDH].

In Section 4 we elaborate on the idea of assigning to each basis element ψ_n a positive weight w_n. We discuss weight-greedy bases and prove a criterion for weight-greedy bases similar to the one for greedy bases (see [KT] and also Theorem 2.1 from Section 2 of this paper).

2. Proof of Theorem 1

The proof goes by induction. We first prove some inequalities in the univariate case. We need some known results. We begin with definitions of unconditional and democratic bases.

Definition 2.1. *A basis $\Psi = \{\psi_k\}_{k=1}^\infty$ of a Banach space X is said to be unconditional if for every choice of signs $\theta = \{\theta_k\}_{k=1}^\infty$, $\theta_k = 1$ or -1, $k = 1, 2, \dots$, the linear operator M_θ defined by $M_\theta(\sum_{k=1}^\infty a_k\psi_k) = \sum_{k=1}^\infty a_k\theta_k\psi_k$ is a bounded operator from X into X.*

Definition 2.2. *We say that a basis $\Psi = \{\psi_k\}_{k=1}^\infty$ is a democratic basis for X if there exists a constant $D := D(X, \Psi)$ such that for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have $\|\sum_{k\in P}\psi_k\| \le D\|\sum_{k\in Q}\psi_k\|$.*

The following theorem has been proved in [KT].

Theorem 2.1. *A normalized basis is greedy if and only if it is unconditional and democratic.*

This theorem gives a characterization of greedy bases. Further investigations ([T2], [CDH], [DKKT], [KerP], [GN], [T4], [KaT]) showed that the concept of greedy bases is very useful in direct and inverse theorems of nonlinear approximation and also in applications in statistics. It has been noticed in [DKKT] that the proof of Theorem 2.1 from [KT] works also for a basis that is not assumed to be normalized (they assumed instead $\inf_n \|\psi_n\| > 0$).

There is a result in functional analysis [KP], [LT] that says that for any unconditional basis $B = (b_k)$ of $L_p([0,1)^d)$, normalized so that $\|b_k\|_p = 1$, there is a subsequence k_j, $j = 1, 2, \dots$, such that (b_{k_j}) satisfies

$$\|\sum_{j=1}^\infty \alpha_{k_j} b_{k_j}\|_p^p \asymp \sum_{j=1}^\infty |\alpha_{k_j}|^p.$$

It follows that for any democratic and unconditional basis B for $L_p([0,1)^d)$, we have

$$\|\sum_{k\in\Lambda} b_k\|_p \asymp (|\Lambda|)^{1/p}$$

with the constants of equivalency depending at most on B and p. For an unconditional, democratic basis B in L_p, then the above results combine to show that

$$C_1 \min_{k\in\Lambda} |a_k| \, (|\Lambda|)^{1/p} \le \| \sum_{k\in\Lambda} a_k b_k \|_p \le C_2 \max_{k\in\Lambda} |a_k| \, (|\Lambda|)^{1/p} \tag{2.1}$$

for any finite set Λ with $C_1, C_2 > 0$ independent of Λ and $\{a_k\}$. This proves Theorem 1 for $d = 1$, $1 < p < \infty$.

We will often use the following known lemma (see [LT,p.73]).

Lemma 2.1. *For any finite collection $\{f_s\}$ of functions in L_p, $1 \le p \le \infty$, we have*

$$(\sum_s \|f_s\|_p^{p_l})^{1/p_l} \le \|(\sum_s |f_s|^2)^{1/2}\|_p \le (\sum_s \|f_s\|_p^{p_u})^{1/p_u} \tag{2.2}$$

with $p_l := \max(2, p)$ and $p_u := \min(2, p)$.

We note that by Theorem 2.1 a greedy basis Ψ is unconditional. It is known that the tensor product of unconditional bases for $L_p([0, 1))$, $1 < p < \infty$, is an unconditional basis for $L_p([0, 1)^d)$. Therefore for any $1 < p < \infty$ and any $\{a_{\mathbf{n}}\}$ we have

$$C_1(p, d)\|(\sum_{\mathbf{n}} |a_{\mathbf{n}}\psi_{\mathbf{n}}|^2)^{1/2}\|_p \le \| \sum_{\mathbf{n}} a_{\mathbf{n}}\psi_{\mathbf{n}}\|_p \le C_2(p, d)\|(\sum_{\mathbf{n}} |a_{\mathbf{n}}\psi_{\mathbf{n}}|^2)^{1/2}\|_p, \tag{2.3}$$

and also for any set of disjoint Λ_j we have

$$C_3(p, d)\|(\sum_j | \sum_{\mathbf{n}\in\Lambda_j} a_{\mathbf{n}}\psi_{\mathbf{n}}|^2)^{1/2}\|_p \le \| \sum_j \sum_{\mathbf{n}\in\Lambda_j} a_{\mathbf{n}}\psi_{\mathbf{n}}\|_p \tag{2.4}$$

$$\le C_4(p, d)\|(\sum_j | \sum_{\mathbf{n}\in\Lambda_j} a_{\mathbf{n}}\psi_{\mathbf{n}}|^2)^{1/2}\|_p.$$

Lemma 2.2. *Let $2 \le p < \infty$ and let Ψ be a greedy basis for $L_p([0, 1))$. Then for any finite Λ, $|\Lambda| = m$, and any coefficients $\{a_k\}$ we have*

$$(\sum_{k\in\Lambda} |a_k|^p)^{1/p} \ll \| \sum_{k\in\Lambda} a_k\psi_k\|_p \ll (\log m)^{1/2-1/p}(\sum_{k\in\Lambda} |a_k|^p)^{1/p}.$$

Proof. The lower estimate follows from (2.3) and Lemma 2.1. We now prove the upper estimate. Let

$$|a_{k_1}| \ge |a_{k_2}| \ge \dots, \quad k_j \in \Lambda, \quad j = 1, 2, \dots, m.$$

For notational convenience we set $a_{k_j} = 0$ for $j > m$. Denoting

$$f_s := \sum_{j=2^s}^{2^{s+1}-1} a_{k_j}\psi_{k_j} \tag{2.5}$$

we get for n such that $2^n \le m < 2^{n+1}$

$$f := \sum_{k\in\Lambda} a_k\psi_k = \sum_{s=0}^{n} f_s. \tag{2.6}$$

By (2.4) and Lemma 2.1 we obtain

$$\|f\|_p \ll (\sum_{s=0}^{n} \|f_s\|_p^2)^{1/2}.$$

Next, by (2.1)

$$\|f_s\|_p \ll |a_{k_{2^s}}|2^{s/p}.$$

Thus

$$\|f\|_p \ll (\sum_{s=0}^{n} |a_{k_{2^s}}|^2 2^{2s/p})^{1/2}.$$

By Hölder's inequality with parameter $p/2$ we continue

$$\le (\sum_{s=0}^{n} |a_{k_{2^s}}|^p 2^s)^{1/p} (\sum_{s=0}^{n} 1)^{(1-2/p)/2} \ll (\log m)^{1/2-1/p} (\sum_{k\in\Lambda} |a_k|^p)^{1/p}.$$

Lemma 2.3. *Let $1 < p \le 2$ and let Ψ be a greedy basis for $L_p([0,1))$. Then for any finite Λ, $|\Lambda| = m$, and any coefficients $\{a_k\}$ we have*

$$(\log m)^{1/2-1/p} (\sum_{k\in\Lambda} |a_k|^p)^{1/p} \ll \|\sum_{k\in\Lambda} a_k\psi_k\|_p \ll (\sum_{k\in\Lambda} |a_k|^p)^{1/p}.$$

Proof. The upper estimate follows from (2.3) and Lemma 2.1. We proceed to the lower estimate. Using the notations (2.5) and (2.6), by (2.4), (2.2), and (2.1) we obtain

$$\|f\|_p \gg (\sum_{s=0}^{n} \|f_s\|_p^2)^{1/2} \gg (\sum_{s=0}^{n} |a_{k_{2^{s+1}}}|^2 2^{2s/p})^{1/2}.$$

Next, by Hölder's inequality with parameter $2/p$ we get

$$\sum_{s=0}^{n} |a_{k_{2^{s+1}}}|^p 2^s \le (\sum_{s=0}^{n} |a_{k_{2^{s+1}}}|^2 2^{2s/p})^{p/2} (n+1)^{1-p/2}.$$

Therefore,

$$\|f\|_p \gg (\sum_{s=0}^{n} |a_{k_{2^{s+1}}}|^p 2^s)^{1/p} n^{1/2-1/p} \gg (\log m)^{1/2-1/p} (\sum_{k\in\Lambda} |a_k|^p)^{1/p}.$$

Proof of Theorem 1. We obtain the lower estimate for $2 \le p < \infty$ and the upper estimate for $1 < p \le 2$ from (2.3) and Lemma 2.1. It remains to prove Theorem 1 in the following cases: $2 \le p < \infty$, the upper estimate and $1 < p \le 2$, the lower estimate. We mentioned above that the assumption that the Ψ is a greedy basis for $L_p([0,1))$ implies that the Ψ^d is an unconditional basis for $L_p([0,1)^d)$. Therefore, it is sufficient to prove Theorem 1 in the particular case of $c_{\mathbf{n}} = 1$, $\mathbf{n} \in \Lambda$. We first prove the upper estimate in the case $2 \le p < \infty$. Let

$$\Lambda_d := \{n_d : \exists \mathbf{k} \in \Lambda \quad \text{with} \quad k_d = n_d\},$$

$$\Lambda(n_d) := \{(k_1, \dots, k_{d-1}) : (k_1, \dots, k_{d-1}, n_d) \in \Lambda\}.$$

Then we have by Lemma 2.2

$$\| \sum_{n_d \in \Lambda_d} \psi_{n_d}(x_d) (\sum_{(n_1,\dots,n_{d-1}) \in \Lambda(n_d)} \psi_{n_1}(x_1) \dots \psi_{n_{d-1}}(x_{d-1})) \|_p^p$$

$$\ll (\log m)^{(1/2-1/p)p} \sum_{n_d \in \Lambda_d} \| \sum_{(n_1,\dots,n_{d-1}) \in \Lambda(n_d)} \psi_{n_1}(x_1) \dots \psi_{n_{d-1}}(x_{d-1})) \|_p^p.$$

We continue by the induction assumption

$$\ll (\log m)^{(1/2-1/p)p} (\sum_{n_d \in \Lambda_d} |\Lambda(n_d)| (\log m)^{(1/2-1/p)p(d-2)})$$

$$= m (\log m)^{(1/2-1/p)(d-1)p}.$$

We proceed to the lower estimate in the case $1 < p \le 2$. By Lemma 2.3 we get

$$\| \sum_{n_d \in \Lambda_d} \psi_{n_d}(x_d) (\sum_{(n_1,\dots,n_{d-1}) \in \Lambda(n_d)} \psi_{n_1}(x_1) \dots \psi_{n_{d-1}}(x_{d-1})) \|_p^p$$

$$\gg (\log m)^{(1/2-1/p)p} \sum_{n_d \in \Lambda_d} \| \sum_{(n_1,\dots,n_{d-1}) \in \Lambda(n_d)} \psi_{n_1}(x_1) \dots \psi_{n_{d-1}}(x_{d-1})) \|_p^p.$$

We continue by the induction assumption

$$\gg (\log m)^{(1/2-1/p)p} (\sum_{n_d \in \Lambda_d} |\Lambda(n_d)| (\log m)^{(1/2-1/p)p(d-2)})$$

$$= m (\log m)^{(1/2-1/p)(d-1)p}.$$

3. Proof of Theorem 2

The lower estimate in the case $2 \le p < \infty$ and the upper estimate in the case $1 < p \le 2$ follow from (2.3) and Lemma 2.1. We first note that the lower estimate in the case $1 < p \le 2$ follows from the upper estimate in the case $2 \le p < \infty$ by the duality argument. Indeed, assume (1.4) has been proved. Let $q \in (1, 2]$. Denote $p := q/(q-1) \in [2, \infty)$. We have

$$\sum_{I\in\Lambda} \| |I|^{-a} h_I \|_q^q = \sum_{I\in\Lambda} |I|^{-aq+1} = \langle \sum_{I\in\Lambda} |I|^{-a} h_I, \sum_{I\in\Lambda} |I|^{-a(q-1)} h_I \rangle$$

$$\le \| \sum_{I\in\Lambda} |I|^{-a} h_I \|_q \| \sum_{I\in\Lambda} |I|^{-a(q-1)} h_I \|_p.$$

Using (1.4) we continue

$$\ll \| \sum_{I\in\Lambda} |I|^{-a} h_I \|_q (\log m)^{(1/2-1/p)(d-1)} (\sum_{I\in\Lambda} \| |I|^{-a(q-1)} h_I \|_p^p)^{1/p}$$

$$= \| \sum_{I\in\Lambda} |I|^{-a} h_I \|_q (\log m)^{(1/2-1/p)(d-1)} (\sum_{I\in\Lambda} |I|^{-aq+1})^{1/p}.$$

This implies the lower estimate in (1.5).

It remains to prove the upper estimate in (1.4). We will carry out the proof by induction. First, consider the univariate case. We have

$$\sum_I \| |I|^{-a} h_I \|_p^p = \sum_I |I|^{-ap+1}$$

and by (2.3)

$$\| \sum_I |I|^{-a} h_I \|_p^p \ll \int_0^1 (\sum_I (|I|^{-a} h_I)^2)^{p/2} = \int_0^1 (\sum_{j=1}^s 2^{2an_j} \chi_{E_j})^{p/2}$$

with some $n_1 < n_2 < \dots < n_s$ and $E_j \subset [0,1]$, $j = 1, \dots, s$. By an analog of Lemma 2.3 from [T1] that follows from its proof we continue

$$\ll \sum_{j=1}^s 2^{2n_j a(p/2)} |E_j| = \sum_{j=1}^s 2^{n_j ap} |E_j| = \sum_I |I|^{-ap+1}.$$

We proceed to the multivariate case. Let

$$\Lambda_d := \{I_d : \exists J \in \Lambda \quad \text{with} \quad J_d = I_d\},$$

$$\Lambda(I_d) := \{(J_1, \dots, J_{d-1}) : (J_1, \dots, J_{d-1}, I_d) \in \Lambda\}.$$

Using the fact ([T1]) that the univariate Haar basis is a greedy basis for $L_p([0,1))$, $1<p<\infty$, we get by Lemma 2.2

$$\|\sum_{I_d\in\Lambda_d}|I_d|^{-a}h_{I_d}(x_d)(\sum_{(J_1,\dots,J_{d-1})\in\Lambda(I_d)}|J_1|^{-a}h_{J_1}(x_1)\dots|J_{d-1}|^{-a}h_{J_{d-1}}(x_{d-1}))\|_p^p$$

$$\ll (\log m)^{(1/2-1/p)p}\sum_{I_d\in\Lambda_d}\||I_d|^{-a}h_{I_d}(x_d)\|_p^p$$

$$\times(\|\sum_{(J_1,\dots,J_{d-1})\in\Lambda(I_d)}|J_1|^{-a}h_{J_1}(x_1)\dots|J_{d-1}|^{-a}h_{J_{d-1}}(x_{d-1})\|_p^p).$$

By the induction assumption we continue

$$\ll (\log m)^{(1/2-1/p)p(d-1)}\sum_{I_d\in\Lambda_d}\||I_d|^{-a}h_{I_d}(x_d)\|_p^p$$

$$\times(\sum_{(J_1,\dots,J_{d-1})\in\Lambda(I_d)}\||J_1|^{-a}h_{J_1}(x_1)\|_p^p\dots\||J_{d-1}|^{-a}h_{J_{d-1}}(x_{d-1})\|_p^p)$$

$$=(\log m)^{(1/2-1/p)p(d-1)}\sum_{I\in\Lambda}\||I|^{-a}h_I\|_p^p.$$

4. Weight-greedy bases

Let Ψ be a basis for X. If $\inf_n\|\psi_n\|>0$ then $c_n(f)\to 0$ as $n\to\infty$, where

$$f=\sum_{n=1}^{\infty}c_n(f)\psi_n.$$

Then we can rearrange the coefficients $\{c_n(f)\}$ in the decreasing way

$$|c_{n_1}(f)|\ge|c_{n_2}(f)|\ge\dots$$

and define the mth greedy approximant as

$$G_m(f,\Psi):=\sum_{k=1}^{m}c_{n_k}(f)\psi_{n_k}. \tag{4.1}$$

In the case $\inf_n\|\psi_n\|=0$ we define $G_m(f,\Psi)$ by (4.1) for f of the form

$$f=\sum_{n\in Y}c_n(f)\psi_n,\quad |Y|<\infty. \tag{4.2}$$

Let a weight sequence $w=\{w_n\}_{n=1}^\infty$, $w_n>0$, be given. For $\Lambda\subset\mathbb{N}$ denote $w(\Lambda):=\sum_{n\in\Lambda}w_n$. For a positive real number $v>0$ define

$$\sigma_v^w(f,\Psi):=\inf_{\{b_n\},\Lambda:w(\Lambda)\le v}\|f-\sum_{n\in\Lambda}b_n\psi_n\|,$$

where Λ are finite.

Definition 4.1. *We call a basis Ψ weight-greedy basis (w-greedy basis) if for any $f \in X$ in the case $\inf_n \|\psi_n\| > 0$ or for any $f \in X$ of the form (4.2) in the case $\inf_n \|\psi_n\| = 0$ we have*

$$\|f - G_m(f, \Psi)\| \le C_G \sigma^w_{w(\Lambda_m)}(f, \Psi),$$

where

$$G_m(f, \Psi) = \sum_{n \in \Lambda_m} c_n(f)\psi_n, \quad |\Lambda_m| = m.$$

Definition 4.2. *We call a basis Ψ weight-democratic basis (w-democratic basis) if for any finite $A, B \subset \mathbb{N}$ such that $w(A) \le w(B)$ we have*

$$\|\sum_{n \in A} \psi_n\| \le C_D \|\sum_{n \in B} \psi_n\|.$$

Theorem 4.1. *A basis Ψ is w-greedy basis if and only if it is unconditional and w-democratic.*

Proof. I. We first prove the implication

unconditional + w-democratic $\quad\Rightarrow\quad$ w-greedy.

Let f be any or of the form (4.2) if $\inf_n \|\psi_n\| = 0$. Consider

$$G_m(f, \Psi) = \sum_{n \in Q} c_n(f)\psi_n =: S_Q(f).$$

We take any finite set $P \subset \mathbb{N}$ satisfying $w(P) \le w(Q)$. Then our assumption $w_n > 0$, $n \in \mathbb{N}$ implies that either $P = Q$ or $Q \setminus P$ is nonempty. Denote

$$\sigma_P(f, \Psi) := \inf_{\{b_n\}} \|f - \sum_{n \in P} b_n \psi_n\|.$$

Then by unconditionality of Ψ we have

$$\|f - S_P(f)\| \le K\sigma_P(f, \Psi). \tag{4.3}$$

This (with $P = Q$) completes the proof in the case $\sigma^w_{w(Q)}(f, \Psi) = \sigma_Q(f, \Psi)$. Suppose that $\sigma^w_{w(Q)}(f, \Psi) < \sigma_Q(f, \Psi)$. Clearly, we now may consider only those P that satisfy the following two conditions

$$w(P) \le w(Q) \quad \text{and} \quad \sigma_P(f, \Psi) < \sigma_Q(f, \Psi).$$

For P satisfying the above conditions we have $Q \setminus P \ne \emptyset$. We estimate

$$\|f - S_Q(f)\| \le \|f - S_P(f)\| + \|S_P(f) - S_Q(f)\|. \tag{4.4}$$

We have

$$S_P(f) - S_Q(f) = S_{P\setminus Q}(f) - S_{Q\setminus P}(f). \tag{4.5}$$

Similarly to (4.3) we get

$$\|S_{Q\setminus P}(f)\| \le K\sigma_P(f,\Psi). \tag{4.6}$$

It remains to estimate $\|S_{P\setminus Q}(f)\|$. We have by unconditionality and w-democracy of Ψ

$$\|S_{P\setminus Q}(f)\| \le 2K \max_{n\in P\setminus Q} |c_n(f)| \|\sum_{n\in P\setminus Q} \psi_n\|$$

$$\le 2KC_D \min_{n\in Q\setminus P} |c_n(f)| \|\sum_{n\in Q\setminus P} \psi_n\| \le 4K^2 C_D \|S_{Q\setminus P}(f)\|. \tag{4.7}$$

Combining (4.3)–(4.7) we complete the proof of part I.

Remark 4.1. Suppose Ψ instead of being w-democratic satisfies the following inequality

$$\|\sum_{n\in A} \psi_n\| \le K(N) \|\sum_{n\in B} \psi_n\|$$

for all $A, B \subset \mathbb{N}$, $w(A) \le w(B) \le N$. Then the above proof gives

$$\|f - G_m(f,\Psi)\| \le CK(w(Q))\sigma^w_{w(Q)}(f,\Psi).$$

II. We now prove the implication
w-greedy $\Rightarrow$ unconditional + w-democratic.
IIa. We begin with the following one
w-greedy $\Rightarrow$ unconditional.
We will prove a little stronger statement.

Lemma 4.1. *Let Ψ be a basis such that for any f of the form (4.2) we have*

$$\|f - G_m(f,\Psi)\| \le C\sigma_\Lambda(f,\Psi),$$

where

$$G_m(f,\Psi) = \sum_{n\in\Lambda} c_n(f)\psi_n.$$

Then Ψ is unconditional.

Proof. It is clear that it is sufficient to prove that there exists a constant C_0 such that for any finite Λ and any f of the form (4.2) we have

$$\|S_\Lambda(f)\| \le C_0\|f\|.$$

Let f and Λ be given and $\Lambda \subset [1, M]$. Consider

$$f_M := S_{[1,M]}(f).$$

Then $\|f_M\| \le C_B \|f\|$. We take a $b > \max_{1\le n\le M} |c_n(f)|$ and define a new function

$$g := f_M - S_\Lambda(f_M) + b \sum_{n\in\Lambda} \psi_n.$$

Then

$$G_m(g, \Psi) = b \sum_{n\in\Lambda} \psi_n, \quad m := |\Lambda|,$$

and

$$\sigma_\Lambda(g, \Psi) \le \|f_M\|.$$

Thus by the assumption

$$\|f_M - S_\Lambda(f_M)\| = \|g - G_m(g, \Psi)\| \le C\sigma_\Lambda(g, \Psi) \le C\|f_M\|.$$

Therefore,

$$\|S_\Lambda(f)\| = \|S_\Lambda(f_M)\| \le C_0\|f\|.$$

IIb. It remains to prove the implication
w-greedy $\quad\Rightarrow\quad$ w-democratic.
First, let $A, B \subset \mathbb{N}$, $w(A) \le w(B)$, be such that $A \cap B = \emptyset$. Consider

$$f := \sum_{n\in A} \psi_n + (1+\epsilon) \sum_{n\in B} \psi_n, \quad \epsilon > 0.$$

Then

$$G_m(f, \Psi) = (1+\epsilon) \sum_{n\in B} \psi_n, \quad m := |B|,$$

and

$$\sigma_A(f, \Psi) \le \|\sum_{n\in B} \psi_n\|(1+\epsilon).$$

Therefore, by the w-greedy assumption we get

$$\|\sum_{n\in A} \psi_n\| \le C(1+\epsilon)\|\sum_{n\in B} \psi_n\|.$$

Let now A, B be arbitrary finite, $w(A) \le w(B)$. Then using unconditionality of Ψ that has already been proven in IIa and the above part of IIb we obtain

$$\|\sum_{n\in A} \psi_n\| \le \|\sum_{n\in A\setminus B} \psi_n\| + \|\sum_{n\in A\cap B} \psi_n\|$$

$$\le C\|\sum_{n\in B\setminus A} \psi_n\| + K\|\sum_{n\in B} \psi_n\| \le C_1\|\sum_{n\in B} \psi_n\|.$$

This completes the proof of Theorem 4.1.

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