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# DECOMPOSITION OF BESOV AND TRIEBEL-LIZORKIN SPACES ON THE SPHERE

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Abstract. A discrete system of almost exponentially localized elements (needlets) on the $n$-dimensional unit sphere $\mathbb{S}^n$ is constructed. It shown that the needlet system can be used for decomposition of Besov and Triebel-Lizorkin spaces on the sphere. As an application of Besov spaces on $\mathbb{S}^n$, a Jackson estimate for nonlinear $m$-term approximation from the needlet system is obtained.

## 1. Introduction

A basic principle in Harmonic analysis is to represent functions or distributions by simple elements (building blocks). The $\varphi$-transform of Frazier and Jawerth [3, 4] and Meyer's wavelets [7] provide such building blocks on $\mathbb{R}^n$. The almost exponential localization and simple structure on the frequency side of the frame elements of Frazier-Jawerth and Meyer's wavelets makes them a universal tool for decomposition of spaces of functions and distributions on $\mathbb{R}^n$.

Our primary goal in this article is to develop similar building blocks on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ ($n \geq 2$). The structure of the function spaces on $\mathbb{S}^n$ is different and more complicated than on $\mathbb{R}^n$ due to the fact that there is no dilation operator on $\mathbb{S}^n$ and the rotation group on $\mathbb{S}^n$ is much more complicated then the shifts in $\mathbb{R}^n$.

The spherical harmonics provide a basic vehicle for representation and analysis of functions on $\mathbb{S}^n$. However, they can be effectively used for decomposition of functions only in $L^2(\mathbb{S}^n)$. If $\mathsf{P}_\nu$ is an appropriately normalized Gegenbauer polynomial of degree $\nu$, then $\mathsf{P}_\nu(\xi \cdot \eta)$ is the kernel of the orthogonal projector onto the space $\mathcal{H}_\nu$ of all spherical harmonics of degree $\nu$ on $\mathbb{S}^n$. Consequently,

$$K_m(\xi \cdot \eta) := \sum_{\nu=0}^{m} \mathsf{P}_\nu(\xi \cdot \eta)$$

is the kernel of the orthogonal projector onto the space of all spherical polynomials of degree $m$. The poor localization of $K_m(\xi \cdot \eta)$ is a major obstacle in using the spherical harmonics for decomposition of function spaces other than $L^2$.

A key fact [10] is that any kernel of the form

$$\Lambda_N(\xi \cdot \eta) = \sum_{\nu=0}^{\infty} \widehat{a}\Big(\frac{\nu}{N}\Big) \mathsf{P}_\nu(\xi \cdot \eta), \tag{1.1}$$

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where $\widehat{a}$ is a compactly supported $C^\infty$ function with $\operatorname{supp} \widehat{a} \subset (0, \infty)$ has nearly exponential localization, namely, for any $k > 0$ there is a constant $c_k > 0$ such that

$$|\Lambda_N(\xi \cdot \eta)| \le \frac{c_k N^n}{(1 + N d(\xi, \eta))^k}, \quad \xi, \eta \in \mathbb{S}^n. \tag{1.2}$$

Here $d(\xi, \eta)$ is the geodesic distance between $\xi, \eta \in \mathbb{S}^n$.

The role of the kernels $\Lambda_N(\xi \cdot \eta)$ is two-fold. First, these kernels enable one to properly define the Besov spaces $B_p^{\alpha q}$ (B-spaces) and the Triebel-Lizorkin spaces $F_{pq}^\alpha$ (F-spaces) on the sphere (in analogy to Peetre's approach [11] to spaces on $\mathbb{R}^n$). Second, they give us a tool for constructing extremely well localized elements (building blocks) on the sphere. The almost exponential localization of our building blocks prompted us to call them *needlets*. The construction of the needlets is based on a Calderón type reproducing formula. Another important ingredient for the construction of a discrete system of needlets on $\mathbb{S}^n$ is the cubature formula from [8, 10]. If we denote the analysis and synthesis needlets by $\varphi_\eta$ and $\psi_\eta$ (see §3), where $\eta$ belongs to a countable set $\mathcal{X}$ of points on the sphere (also an index set), then every distribution $f$ on $\mathbb{S}^n$ ($f \in \mathcal{S}'(\mathbb{S}^n)$) has the representation

$$f = \sum_{\eta \in \mathcal{X}} \langle f, \varphi_\eta \rangle \psi_\eta.$$

The needlets enjoy the following properties which make them a handy tool on the sphere:

(a) Each needlet is a zonal polynomial, i.e. a function of the form $g(\xi \cdot \eta)$, where $g$ is a univariate algebraic polynomial.

(b) Each needlet is "compactly supported and infinitely smooth" on the frequency side, namely, it is of the form (1.1) with $\widehat{a}$ a compactly supported $C^\infty$ function.

(c) Each needlet $\varphi_\eta$ or $\psi_\eta$ is localized around a certain point (center) $\eta \in \mathcal{X}$ and is rapidly decaying away from this point (with rate as in (1.2)).

(d) The needlets are semi-orthogonal, namely, every two of them which are from levels at least two levels apart are orthogonal.

Although the needlets do not form a basis, they behave like a basis. In [10], among other things, it is shown that when $\varphi_\eta = \psi_\eta$ the needlet system $\{\psi_\eta\}_{\eta \in \mathcal{X}}$ is a tight frame for $L^2(\mathbb{S}^n)$.

In this article we show that the needlet system can be applied to obtain norm characterizations of function spaces covered by the Littlewood-Paley theory on $\mathbb{S}^n$, in general, Besov and Triebel-Lizorkin spaces. These include the $L^p(\mathbb{S}^n)$ spaces, $1 < p < \infty$, the Hardy spaces $H^p(\mathbb{S}^n)$ spaces, $0 < p \le 1$, and the Riesz potential spaces. We have the following characterization of the Triebel-Lizorkin space $F_p^{\alpha q}$ on $\mathbb{S}^n$, where $\alpha \in \mathbb{R}$ and $0 < p < \infty$, $0 < q \le \infty$:

$$\|f\|_{F_p^{\alpha q}} \approx \Big\| \Big( \sum_{\eta \in \mathcal{X}} \Big[ |G_\eta|^{-\alpha/n - 1/2} |\langle f, \varphi_\eta \rangle| \mathbb{1}_{G_\eta}(\cdot) \Big]^q \Big)^{1/q} \Big\|_{L^p}, \quad f \in F_p^{\alpha q}.$$

Here $G_\eta$ is a spherical cap on $\mathbb{S}^n$ centered at $\eta \in \mathcal{X}$ of geodesic radius $c2^{-jn}$ if $\eta \in \mathcal{X}_j$ the $j$th level in $\mathcal{X}$ ($\mathcal{X} = \bigcup_{j \ge 0} \mathcal{X}_j$) (§4). For the Besov space $B_p^{\alpha q}$ on $\mathbb{S}^n$, where $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, we have

$$\|f\|_{B_p^{\alpha q}} \approx \Big( \sum_{j=0}^{\infty} \Big[ 2^{j(\alpha + n/2 - n/p)} \Big( \sum_{\eta \in \mathcal{X}_j} |\langle f, \varphi_\eta \rangle|^p \Big)^{1/p} \Big]^q \Big)^{1/q}, \quad f \in B_p^{\alpha q},$$

where $\{\varphi_\eta\}_{\eta\in\mathcal{X}_j}$ are the needlets of level $j$ (§5). These results are analogous to the fundamental results of Frazier and Jawerth [3, 4] (see also [5]).

Atomic and molecular decompositions of Besov and Triebel-Lizorkin spaces on the sphere can be developed similarly as on $\mathbb{R}^n$, where the approach of Frazier-Jawerth [3, 4] can be utilized. These can be used in showing that the Besov and Triebel-Lizorkin spaces on the sphere used in this paper are the same as the ones induced by the more general definition of Besov and Triebel-Lizorkin spaces on manifolds given in e.g. [19]. We do not go in this direction since the sole purpose of this paper is to develop the needlet system. An important motivation for this work is the application of Besov spaces on the sphere to nonlinear $m$-term approximation from needlets; we prove a Jackson estimate for approximation in $L_p$, $0 < p < \infty$ (§6).

The organization of the paper is as follows. Section 2 contains some preliminaries, including localized kernels on the sphere, cubature formulae, maximal inequalities and basics of distributions on the sphere. In §3, we introduce the needlet system and give some of its basic properties. In §4, we show that the needlets can be used for characterization of the Triebel-Lizorkin spaces on the sphere. In §5, we show that the Besov spaces on the sphere can be characterized via the needlet system. In §6, we apply Besov spaces to nonlinear $m$-term approximation from needlets. Section 7 is an appendix, where we place the proofs of some lemmas from previous sections.

Throughout the article, we use the following notation: $\Pi_m$ denotes the set of all univariate algebraic polynomials of degree $\le m$ and $\Pi_m(\mathbb{S}^n)$ denotes the set of all spherical polynomials of degree $\le m$. For any set $E \subset \mathbb{S}^n$, $\mathbb{1}_E$ denotes the characteristic function of $E$ and $|E|$ denotes the Lebesgue measure of $E$. The geodesic distance on $\mathbb{S}^n$ is denoted by $d(\xi,\eta)$, i.e. $d(\xi,\eta) := \arccos \xi\cdot\eta$, where $\xi\cdot\eta$ denotes the inner product of $\xi,\eta \in \mathbb{S}^n$. We use the notation $B_\eta(r) := \{\xi \in \mathbb{S}^n : d(\xi,\eta) \le r\}$. Positive constants are denoted by $c, c_1, c^*, \dots$ (they may vary at every occurrence), $A \approx B$ means $c_1 A \le B \le c_2 B$, and $A := B$ stands for "$A$ is by definition equal to $B$".

## 2. Preliminaries

2.1. **Localized polynomial kernels on the sphere.** Let $\mathsf{P}_\nu$ be the orthogonal projector onto the subspace $\mathcal{H}_\nu$ of all spherical harmonics of order $\nu$ on $\mathbb{S}^n$. As is well known the kernel of $\mathsf{P}_\nu$ is given by

$$\mathsf{P}_\nu(\xi\cdot\eta) = \frac{\nu+\lambda}{\lambda\omega_n} P_\nu^\lambda(\xi\cdot\eta), \quad \lambda := \frac{n-1}{2}, \tag{2.1}$$

where $\omega_n$ denotes the hypersurface area of $\mathbb{S}^n$. Here $P_\nu^\lambda$ is the Gegenbauer polynomial of degree $\nu$ normalized with $P_\nu^\lambda(1) = \binom{\nu+2\lambda-1}{\nu}$ [17, p. 81, (4.7.1)]. We refer the reader to [9, 15] for the basics of spherical harmonics.

Consider now a kernel (polynomial) of the form

$$\Lambda_N(x) = \sum_{\nu=0}^{\infty} \widehat{a}\left(\frac{\nu}{N}\right) \mathsf{P}_\nu(x) \tag{2.2}$$

with "smoothing function" $\widehat{a}$ obeying the following definition:

**Definition 2.1.** *A function $\widehat{a}$ is said to be admissible if $\widehat{a} \in C^\infty[0,\infty)$ and $\widehat{a}$ satisfies one of the following two conditions:*

(a) $\operatorname{supp} \widehat{a} \subset [0,2]$, $\widehat{a}(t) = 1$ *on* $[0,1]$, *and* $0 \le \widehat{a}(t) \le 1$ *on* $[1,2]$; *or*
(b) $\operatorname{supp} \widehat{a} \subset [1/2, 2]$.

Our development in this article heavily relies on the fundamental fact that every polynomial $\Lambda_N$ as above has nearly exponential localization around zero.

**Theorem 2.2.** [10] *Let $\widehat{a}$ be admissible. Then for every $k > 0$ and $r \ge 0$ there exists a constant $c_{k,r} > 0$ depending only on $k$, $r$, $n$ and $\widehat{a}$ such that*

$$\Big|\frac{d^r}{dx^r}\Lambda_N(\cos\theta)\Big| \le c_{k,r}\frac{N^{n+2r}}{(1+N\theta)^k}, \quad \theta \in [0,\pi]. \tag{2.3}$$

*The dependence of $c_k$ on $\widehat{a}$ is of the form $c_k = c(k,r,\lambda)\max_{0\le m\le k}\|\widehat{a}^{(m)}\|_{L^\infty}$.*

This estimate is proved in [10]. It also follows from the general result in [6] on the spectral properties of elliptic operators. It was simultaneously extended for Jacobi polynomials in [1] and [13]. For reader's convenience we next state the result for Jacobi polynomials. Denote

$$L_N(x) := c^\diamond \sum_{j=0}^\infty \widehat{a}\Big(\frac{j}{N}\Big)\frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)}P_j^{(\alpha,\beta)}(x), \tag{2.4}$$

where $c^\diamond := \Gamma(\beta+1)/\Gamma(\alpha+\beta+2)$ and $P_j^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials [17, Chapter IV].

**Theorem 2.3.** [1, 13] *Let $\widehat{a}$ be admissible and assume that $\alpha \ge \beta > -1/2$. Then for every $k > 0$ and $r \ge 0$ there exists a constant $c_k > 0$ depending only on $k$, $r$, $\alpha$, $\beta$, and $\widehat{a}$ such that*

$$\Big|\frac{d^r}{dx^r}L_N(\cos\theta)\Big| \le c_{k,r}\frac{N^{2\alpha+2r+2}}{(1+N\theta)^k}, \quad \theta \in [0,\pi]. \tag{2.5}$$

Since [17, (4.7.1), p. 81]

$$P_\nu^\lambda(x) = \frac{\Gamma(\lambda-1/2)}{\Gamma(2\lambda)}\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+\lambda+1/2)}P_\nu^{(\lambda-1/2,\lambda-1/2)}(x),$$

then with $\alpha = \beta = \lambda - 1/2$ we have $\Lambda_N = \omega_n^{-1}L_N$. Consequently, (2.5) yields (2.3).

**Reproducing kernels.** Assuming that $\widehat{a}$ is admissible of type (a), we define

$$\mathcal{K}_N := \sum_{\nu=0}^\infty \widehat{a}\Big(\frac{\nu}{N}\Big)\mathsf{P}_\nu, \ N = 1, 2, \dots. \tag{2.6}$$

We next give some basic properties of the kernels $\mathcal{K}_N(\xi\cdot\eta)$. We begin with two definitions.
**Nonstandard convolution on $\mathbb{S}^n$**: For functions $\Phi \in L^\infty[-1,1]$ and $f \in L^1(\mathbb{S}^n)$, we write

$$\Phi * f(\xi) := \int_{\mathbb{S}^n}\Phi(\xi\cdot\sigma)f(\sigma)\,d\mu(\sigma). \tag{2.7}$$

**Best polynomial approximation on $\mathbb{S}^n$**: We let $E_m(f)_p$ denote the best approximation of $f \in L^p$ from $\Pi_m(\mathbb{S}^n)$, i.e.

$$E_m(f)_p := \inf_{g\in\Pi_m(\mathbb{S}^n)}\|f-g\|_{L^p}. \tag{2.8}$$

**Lemma 2.4.** *(a) $\mathcal{K}_N$ is a polynomial of degree $< 2N$ and $\mathcal{K}_N$ defines a reproducing kernel for $\Pi_N(\mathbb{S}^n)$, that is, $\mathcal{K}_N * g = g$ for $g \in \Pi_N(\mathbb{S}^n)$.*

*(b) For every $k > 0$ and $r \geq 0$ there exists a constant $c_{k,r} > 0$ such that*

$$\left|\frac{d^r}{dx^r}\mathcal{K}_N(\xi\cdot\eta)\right| \leq \frac{c_{k,r}N^{n+2r}}{(1+Nd(\xi\cdot\eta))^k}, \quad \xi,\eta \in \mathbb{S}^n. \tag{2.9}$$

*(c) For any $f \in L^p(\mathbb{S}^n)$, $1 \leq p \leq \infty$, we have $\mathcal{K}_N * f \in \Pi_{2N}(\mathbb{S}^n)$,*

$$\|\mathcal{K}_N * f\|_{L^p} \leq c\|f\|_{L^p}, \text{ and } \|f - \mathcal{K}_N * f\|_{L^p} \leq cE_N(f)_p. \tag{2.10}$$

**Proof.** Part (a) of the lemma is obvious since $\widehat{a}(\frac{\nu}{N}) = 1$ for $0 \leq \nu \leq N$. Estimate (2.9) follows by (2.3). The first estimate in (2.10) is immediate from (2.9) when $p = 1$ and $p = \infty$; the general case follows by interpolation. The second estimate in (2.10) is a consequence of the first one and (a). □

An easy consequence of the above lemma is the following classical inequality (Nikolski inequality).

**Proposition 2.5.** *For $0 < q \leq p \leq \infty$ and $g \in \Pi_N(\mathbb{S}^n)$,*

$$\|g\|_{L^p} \leq cN^{n(1/q-1/p)}\|g\|_{L^q}. \tag{2.11}$$

For future use we next give one more property of $\mathcal{K}_N$.

**Lemma 2.6.** *Suppose $\omega, \eta, \xi_1, \xi_2 \in \mathbb{S}^n$ and $d(\xi_j, \eta) \leq c^*N^{-1}$, $j = 1, 2$. Then for any $k > 0$ there exists a constant $c_k$ such that*

$$|\mathcal{K}_N(\omega\cdot\xi_1) - \mathcal{K}_N(\omega\cdot\xi_2)| \leq c_k\frac{d(\xi_1,\xi_2)N^{n+1}}{(1+Nd(\omega,\eta))^k}. \tag{2.12}$$

**Proof.** Let $G^* := \{\xi \in \mathbb{S}^n : d(\xi,\eta) \leq c^*N^{-1}\}$. Evidently,

$$|\mathcal{K}_N(\omega\cdot\xi_1) - \mathcal{K}_N(\omega\cdot\xi_2)| \leq \sup_{\xi\in G^*}\left|\frac{d}{dx}\mathcal{K}_N(\omega\cdot\xi)\right||\omega\cdot\xi_1 - \omega\cdot\xi_2|. \tag{2.13}$$

From Lemma 2.4, (b) with $k$ replaced by $k+1$, we have

$$\left|\frac{d}{dx}\mathcal{K}_N(\omega\cdot\xi)\right| \leq \frac{c_{k+1,1}N^{n+2}}{(1+Nd(\omega,\xi))^{k+1}}$$

and hence

$$\sup_{\xi\in G^*}\left|\frac{d}{dx}\mathcal{K}_N(\omega\cdot\xi)\right| \leq \frac{cN^{(n+2)}}{(1+Nd(\omega,\eta))^{k+1}}. \tag{2.14}$$

To estimate $|\omega\cdot\xi_1 - \omega\cdot\xi_2|$, we let $\theta_1 := d(\omega,\xi_1)$ and $\theta_2 := d(\omega,\xi_2)$. Then

$$\begin{aligned}
|\omega\cdot\xi_1 - \omega\cdot\xi_2| &= |\cos\theta_1 - \cos\theta_2| \\
&= 2|\sin(\theta_1-\theta_2)/2||\sin(\theta_1+\theta_2)/2| \\
&\leq (1/2)|\theta_1-\theta_2||\theta_1+\theta_2| \\
&= (1/2)|d(\omega,\xi_1) - d(\omega,\xi_2)|(d(\omega,\xi_1) + d(\omega,\xi_2))
\end{aligned}$$

and hence

$$|\omega\cdot\xi_1 - \omega\cdot\xi_2| \leq d(\xi_1,\xi_2)\max\{d(\omega,\xi_1), d(\omega,\xi_2)\} \leq cd(\xi_1,\xi_2)[d(\omega,\eta) + N^{-1}].$$

Substituting this and (2.14) in (2.13), we obtain

$$|\mathcal{K}_N(\omega\cdot\xi_1) - \mathcal{K}_N(\omega\cdot\xi_2)| \leq \frac{cd(\xi_1,\xi_2)[d(\omega,\eta)+N^{-1}]N^{n+2}}{(1+Nd(\omega,\eta))^{k+1}} \leq \frac{cd(\xi_1,\xi_2)N^{n+1}}{(1+Nd(\omega,\eta))^k},$$

which completes the proof. □

2.2. **Cubature formula on $\mathbb{S}^n$.** For the construction of our discrete systems of building blocks (needlets) we will need a cubature formula on $\mathbb{S}^n$ exact for all spherical polynomials of degree $N$. One of the main difficulties in constructing cubature formulae on the sphere is the lack of uniformly distributed points on $\mathbb{S}^n$. We will use as a substitute sets of almost equally distributed points which we describe in the following.

**Lemma 2.7.** *For any $0 < \varepsilon \le 2\pi$ there exists a partition $\mathcal{R}_\varepsilon$ of $\mathbb{S}^n$ consisting of spherical simplices and a set $\mathcal{X}_\varepsilon \subset \mathbb{S}^n$ (consisting of their "centers") with the properties:*

(i) $\mathbb{S}^n = \bigcup_{R\in\mathcal{R}_\varepsilon} R$ *and the sets in $\mathcal{R}_\varepsilon$ do not overlap ($R_1^\circ \cap R_2^\circ = \emptyset$ if $R_1 \ne R_2$).*

(ii) *For each $R \in \mathcal{R}_\varepsilon$ there is a unique $\eta \in \mathcal{X}_\varepsilon$ such that $B_\eta(c^*\varepsilon) \subset R \subset B_\eta(\varepsilon)$, where $B_\eta(r) := \{\xi \in \mathbb{S}^n : d(\xi,\eta) \le r\}$.*

(iii) $\#\mathcal{X}_\varepsilon = \#\mathcal{R}_\varepsilon \le c^{**}\varepsilon^{-n}$.

*Here $c^*$ and $c^{**}$ are constants depending only on $n$.*

For the proof of this simple lemma, see [10].

**Definition.** A set $\mathcal{X}_\varepsilon \subset \mathbb{S}^n$ which along with an associated partition $\mathcal{R}_\varepsilon$ of $\mathbb{S}^n$ has the properties of the sets $\mathcal{X}_\varepsilon$ and $\mathcal{R}_\varepsilon$ of Lemma 2.7 will be called a *set of almost uniformly $\varepsilon$-distributed points* on $\mathbb{S}^n$.

**Theorem 2.8.** *There exists a constant $c^\diamond > 0$ (depending only on $n$) such that for any $N \ge 1$ and a set $\mathcal{X}_\varepsilon$ of almost uniformly $\varepsilon$-distributed points on $\mathbb{S}^n$ with $\varepsilon := c^\diamond/N$, there exist positive coefficients $\{c_\eta\}_{\eta\in\mathcal{X}_\varepsilon}$ such that the cubature formula*

$$\int_{\mathbb{S}^n} f(\xi)\, d\mu(\xi) \sim \sum_{\eta\in\mathcal{X}_\varepsilon} c_\eta f(\eta)$$

*is exact for all spherical polynomials of degree $\le N$. In addition, $c_\eta \approx N^{-n}$ with constants of equivalence depending only on $n$.*

This theorem is given in [10] and is a slightly improved version of the result from [8].

For the construction of our needlets (§3), we will use the following result which is an immediate consequence of Lemma 2.7 and Theorem 2.8.

**Corollary 2.9.** *There exists a sequence $\{\mathcal{X}_j\}_{j=0}^\infty$ of sets of almost uniformly $\varepsilon_j$-distributed points on $\mathbb{S}^n$ ($\mathcal{X}_j := \mathcal{X}_{\varepsilon_j}$) with $\varepsilon_j := c^\diamond 2^{-j-2}$ and there exist nonnegative coefficients $\{c_\eta\}_{\eta\in\mathcal{X}_j}$ such that the cubature*

$$\int_{\mathbb{S}^n} f(\xi)\, d\mu(\xi) \sim \sum_{\eta\in\mathcal{X}_j} c_\eta f(\eta) \tag{2.15}$$

*is exact for all spherical polynomials of degree $\le 2^{j+2}$. Moreover, $c_\eta \approx 2^{-jn}$ and $\#\mathcal{X}_j \approx 2^{jn}$ with constants depending only on $n$.*

*Furthermore, there exists a constant $c_1 = c_1(n)$ such that if we denote*

$$G_\eta := B_\eta(c_1 2^{-j}) = \{\xi \in \mathbb{S}^n : d(\xi,\eta) \le c_1 2^{-j}\}, \quad \eta \in \mathcal{X}_j, \tag{2.16}$$

*then $\mathbb{S}^n \subset \bigcup_{\eta\in\mathcal{X}_j} G_\eta$ and only finitely many ($\le c(n)$) sets $\{G_\eta\}_{\eta\in\mathcal{X}_j}$ may intersect at a time.*

*Also, we denote*

$$\mathcal{X} := \bigcup_{j=0}^{\infty} \mathcal{X}_j, \tag{2.17}$$

*and we will assume that* $\mathcal{X}$ *consists of distinct points* ($\mathcal{X}$ *will be used as an index set*).

2.3. **Maximal inequality.** We denote by $\mathcal{G}$ the set of all spherical caps on $\mathbb{S}^n$, i.e. $G \in \mathcal{G}$ if $G$ is of the form: $G := \{\xi \in \mathbb{S}^n : d(\xi, \eta) \le \rho\}$ with $\eta \in \mathbb{S}^n$ and $\rho > 0$.

Let $\mathcal{M}_s$ be the maximal operator, defined by

$$\mathcal{M}_s f(\xi) := \sup_{G \in \mathcal{G}\colon \xi \in G} \left( \frac{1}{|G|} \int_G |f(\omega)|^s \, d\mu(\omega) \right)^{1/s}, \quad \xi \in \mathbb{S}^n. \tag{2.18}$$

We will need the Fefferman-Stein vector-valued maximal inequality (see [14]): If $0 < p < \infty$, $0 < q \le \infty$, and $0 < s < \min\{p, q\}$, then for any sequence of functions $f_1, f_2, \dots$ on $\mathbb{S}^n$

$$\Big\| \Big( \sum_{j=1}^{\infty} [\mathcal{M}_s f_j(\cdot)]^q \Big)^{1/q} \Big\|_{L^p} \le c \Big\| \Big( \sum_{j=1}^{\infty} |f_j(\cdot)|^q \Big)^{1/q} \Big\|_{L^p} \tag{2.19}$$

where $c = c(p, q, s, n)$.

For later use, we record the following estimate of $\mathcal{M}_s \mathbb{1}_{G_\eta}(\xi)$. For $s > 0$ and $\eta \in \mathcal{X}_j$ $(j \ge 0)$, we have

$$\mathcal{M}_s \mathbb{1}_{G_\eta}(\xi) \approx \frac{1}{(1 + 2^j d(\xi, \eta))^{n/s}}, \quad \xi \in \mathbb{S}^n, \tag{2.20}$$

with constants of equivalence depending only on $s$, $n$, and $c_1$ (from (2.16)). This equivalence follows by straightforward calculations.

2.4. **Distributions on $\mathbb{S}^n$.** We will use the standard notation:

$$D^\alpha := \partial^{|\alpha|} / \partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}, \quad \text{where} \quad \alpha = (\alpha_1, \dots, \alpha_{n+1}), \quad |\alpha| := \alpha_1 + \dots + \alpha_{n+1}.$$

For a function $\phi$ defined on $\mathbb{S}^n$, we denote by $E\phi$ its extension to $\mathbb{R}^{n+1} \setminus \{\emptyset\}$ defined by $E\phi(x) := \phi(x/|x|)$ and then

$$D^\alpha \phi := D^\alpha (E\phi)|_{\mathbb{S}^n}.$$

Let $\mathcal{S} := C^\infty(\mathbb{S}^n)$ be the set of all test functions on the sphere. The topology on $\mathcal{S}$ is defined by the semi-norms

$$P_r(\phi) := \sum_{|\alpha| = r} \|D^\alpha \phi\|_\infty, \quad r = 0, 1, \dots. \tag{2.21}$$

It is well-known that the spherical harmonics of degree $\nu$ are eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ ($\Delta_{\mathbb{S}^n} f := \Delta E f|_{\mathbb{S}^n}$, $\Delta := \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_{n+1}^2$) with eigenvalues $-\nu(\nu + n - 1)$. The topology in $\mathcal{S}$ can be equivalently defined by the semi-norms

$$P_r^*(\phi) := \|\Delta_{\mathbb{S}^n}^r \phi\|_\infty, \quad r = 0, 1, \dots. \tag{2.22}$$

The space $\mathcal{S}' := \mathcal{S}'(\mathbb{S}^n)$ of all distributions on $\mathbb{S}^n$ is defined as the space of all continuous linear functionals on $\mathcal{S}$ ($\mathcal{S}'$ is the dual of $\mathcal{S}$). The pairing of $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ will be denoted by $\langle f, \phi \rangle := f(\overline{\phi})$, which is consistent with the inner product $\langle f, g \rangle := \int_{\mathbb{S}^n} f \overline{g} d\mu$ in $L^2(\mathbb{S}^n)$.

We now extend the nonstandard convolution defined in (2.7):

**Definition.** If $f \in \mathcal{S}'$ and $\Phi$ is a univariate function such that $\Phi(\xi \cdot \eta)$ belongs to $\mathcal{S}$ as a function of $\eta$ (or $\xi$), we define $\Phi * f$ by the identity

$$\Phi * f(\xi) := \langle f, \overline{\Phi(\xi \cdot \bullet)} \rangle, \tag{2.23}$$

where on the right $f$ acts on $\Phi(\xi \cdot \eta)$ as a function of $\eta$.

**Lemma 2.10.** *(a) If $f \in \mathcal{S}'$ and $\Phi(\xi \cdot \bullet) \in \mathcal{S}$, then $\Phi * f \in \mathcal{S}$.*

*(b) If $f \in \mathcal{S}'$, $\Phi(\xi \cdot \bullet) \in \mathcal{S}$, and $\phi \in \mathcal{S}$, then $\langle \Phi * f, \phi \rangle = \langle f, \overline{\Phi} * \phi \rangle$.*

*(c) If $f \in \mathcal{S}'$ and $\Phi(\xi \cdot \bullet), \Psi(\xi \cdot \bullet) \in \mathcal{S}$, then*

$$\Psi * \overline{\Phi} * f(\xi) = \langle \Psi(\xi \cdot \bullet), \Phi(\bullet \cdot \bullet) \rangle * f. \tag{2.24}$$

*(d) If $f \in \mathcal{S}'$ and $\Phi \in \Pi_m$, then $\Phi * f \in \Pi_m(\mathbb{S}^n)$, and moreover, if $\Phi \in span\{P_{k+1}^\lambda, \dots, P_n^\lambda\}$, then $\Phi * f \in \Pi_m(\mathbb{S}^n) \ominus \Pi_k(\mathbb{S}^n)$.*

This lemma follows by standard arguments and the proof will be omitted.

As was mentioned above, $\mathsf{P}_k$ from (2.1) defines the kernel of the orthogonal projector onto $\mathcal{H}_k$ and, therefore, $f = \sum_{k=0}^\infty \mathsf{P}_k * f$ for $f \in L^2(\mathbb{S}^n)$.

It is well known that $\phi \in C^\infty(\mathbb{S}^n)$ if and only if

$$\|\mathsf{P}_\nu * \phi\|_{L^2} \le c_k(\nu+1)^{-k}, \quad \nu = 0, 1, \dots, \quad \text{for all } k.$$

Consequently, the topology in $\mathcal{S}$ can be equivalently defined by the norms

$$P_r^{**}(\phi) := \sum_{\nu=0}^\infty (\nu+1)^r \|\mathsf{P}_\nu * \phi\|_{L^2}, \quad r = 0, 1, \dots. \tag{2.25}$$

2.5. **Semi-discrete decomposition of $\mathcal{S}'$.** Define

$$\Phi_0 := \mathsf{P}_0 \text{ and } \Phi_j := \sum_{\nu=0}^\infty \widehat{a}\Big(\frac{\nu}{2^{j-1}}\Big)\mathsf{P}_\nu, \quad j = 1, 2, \dots, \tag{2.26}$$

where $\widehat{a}$ satisfies the conditions:

$$\widehat{a} \in C^\infty[0,\infty), \quad \operatorname{supp} \widehat{a} \subset [1/2, 2], \tag{2.27}$$

$$|\widehat{a}(t)| > c > 0 \quad \text{if } t \in [3/5, 5/3], \tag{2.28}$$

$$\widehat{a}(t) + \widehat{a}(2t) = 1 \quad \text{if } t \in [1/2, 1]. \tag{2.29}$$

Hence

$$\sum_{\nu=0}^\infty \widehat{a}(2^{-\nu} t) = 1, \quad t \in [1, \infty). \tag{2.30}$$

It is easy to construct a function $\widehat{a}$ satisfying (2.27)-(2.29). Indeed, it is well-known that there is a function $g \in C^\infty(\mathbb{R})$ such that supp $g \subset [1/2, 2]$ and $g(t) > 0$ on $(1/2, 2)$. Then the function

$$\widehat{a}(t) := \frac{g(t)}{g(2t) + g(t) + g(t/2)}, \quad t \in \mathbb{R},$$

where $0/0 := 0$, satisfies (2.27)-(2.29).

By Theorem 2.2, $\Phi_j$ has the following localization:

$$|\Phi_j(\xi \cdot \eta)| \le \frac{c_k 2^{jn}}{(1 + 2^j d(\xi, \eta))^k}, \quad \xi, \eta \in \mathbb{S}^n, \quad \forall k. \tag{2.31}$$

**Lemma 2.11.** (a) *If $f \in \mathcal{S}'$, then*

$$f = \sum_{j=0}^{\infty} \Phi_j * f \quad \text{in } \mathcal{S}'. \tag{2.32}$$

(b) *If $f \in L^p(\mathbb{S}^n)$, $1 \le p \le \infty$, then* (2.32) *holds in $L^p$.*

**Proof.** By (2.30) it follows that if $\phi \in \mathcal{S}$, then $\phi = \sum_{j=0}^{\infty} \Phi_j * \phi$ in $\mathcal{S}$. Using this (2.32) follows readily.

For the proof of part (b), we observe that $\sum_{j=0}^{\ell} \Phi_j * \phi = \mathcal{K}_\ell * \phi$ with $\mathcal{K}_\ell := \sum_{j=0}^{\ell} \Phi_j$, where $\mathcal{K}_\ell$ is a kernel with properties similar to the properties of $\mathcal{K}_N$ in Lemma 2.4, because of (3.4). Consequently, $\sum_{j=0}^{\ell} \Phi_j * f \to f$ in $L^p$, $1 \le p \le \infty$. $\square$

## 3. Needlets: Definition and properties

Let $\widehat{a}$, $\widehat{b}$ satisfy the conditions:

$$\widehat{a}, \widehat{b} \in C^\infty(\mathbb{R}), \quad \operatorname{supp} \widehat{a}, \operatorname{supp} \widehat{b} \subset [1/2, 2], \tag{3.1}$$

$$|\widehat{a}(t)|, |\widehat{b}(t)| > c > 0 \quad \text{if } t \in [3/5, 5/3], \tag{3.2}$$

$$\overline{\widehat{a}(t)}\, \widehat{b}(t) + \overline{\widehat{a}(2t)}\, \widehat{b}(2t) = 1 \quad \text{if } t \in [1/2, 1]. \tag{3.3}$$

Consequently,

$$\sum_{\nu=0}^{\infty} \overline{\widehat{a}(2^{-\nu}t)}\, \widehat{b}(2^{-\nu}t) = 1, \quad t \in [1, \infty). \tag{3.4}$$

**Lemma 3.1.** (*a*) *If $\widehat{a}$ satisfies* (3.1)$-$(3.2)*, then there exists $\widehat{b}$ satisfying* (3.1)$-$(3.2) *such that* (3.3) *holds true.*

(*b*) *There exists a function $\widehat{a} \ge 0$ satisfying* (3.1) $-$ (3.2) *such that*

$$\widehat{a}^2(t) + \widehat{a}^2(2t) = 1, \quad t \in [1/2, 1], \tag{3.5}$$

*and hence*

$$\sum_{\nu=0}^{\infty} \widehat{a}^2(2^{-\nu}t) = 1, \quad t \in [1, \infty). \tag{3.6}$$

This is an easy and well known lemma. There is a clear connection between the $\hat{a}$'s, $\hat{b}$'s and wavelet masks. In particular, one can use Daubechies wavelet masks to construct a variety of $\hat{a}$'s and $\hat{b}$'s that have interesting properties (see [10]).

Assuming that $\widehat{a}$, $\widehat{b}$ satisfy (3.1)-(3.3), we define

$$\Phi_0 := \mathsf{P}_0 \quad \text{and} \quad \Phi_j := \sum_{\nu=0}^{\infty} \widehat{a}\Big(\frac{\nu}{2^{j-1}}\Big) \mathsf{P}_\nu, \quad j = 1, 2, \dots, \tag{3.7}$$

and

$$\Psi_0 := \mathsf{P}_0 \quad \text{and} \quad \Psi_j := \sum_{\nu=0}^{\infty} \widehat{b}\Big(\frac{\nu}{2^{j-1}}\Big) \mathsf{P}_\nu, \quad j = 1, 2, \dots, \tag{3.8}$$

Further, for $\eta \in \mathcal{X}_j$, we set

$$\varphi_\eta(\xi) := \sqrt{c_\eta}\Phi_j(\xi \cdot \eta) \text{ and } \psi_\eta(\xi) := \sqrt{c_\eta}\Psi_j(\xi \cdot \eta). \tag{3.9}$$

Here $\mathcal{X}_j$ is the set of the nodes and the $c_\eta$'s are the coefficients of the cubature formula from (2.15). Note that $c_\eta \approx 2^{-jn}$. Recall that $\mathcal{X} := \bigcup_{j=0}^\infty \mathcal{X}_j$, which will be used as an index set (see (2.17)).

The functions $\Phi_j$ and $\Psi_j$ inherit all properties of the $\Phi_j$'s defined in (2.26). In particular (see (2.31) and also Theorem 2.2),

$$|\Phi_j(\xi\cdot\eta)|, |\Psi_j(\xi\cdot\eta)| \le \frac{c_k 2^{jn}}{(1+2^j d(\xi,\eta))^k}, \quad \xi,\eta\in\mathbb{S}^n, \quad \forall k, \tag{3.10}$$

and hence

$$|\varphi_\eta(\xi)|, |\psi_\eta(\xi)| \le \frac{c_k 2^{jn/2}}{(1+2^j d(\xi,\eta))^k}, \quad \xi\in\mathbb{S}^n, \quad \forall k. \tag{3.11}$$

Recall that $d(\xi,\eta)$ is the geodesic distance between $\xi$ and $\eta$.

The tremendous localization of $\varphi_\eta$ and $\psi_\eta$ is the reason for calling them *needlets.* Moreover, according to their further roles, we will call $\{\varphi_\eta\}$ *analysis needlets* and $\{\psi_\eta\}$ *synthesis needlets.*

We will need estimates for the norms of the needlets. We have for $0 < p \le \infty$,

$$\|\Phi_j(\bullet\cdot\eta)\|_{L^p} \approx \|\Psi_j(\bullet\cdot\eta)\|_{L^p} \approx 2^{jn(1-1/p)} \text{ and } \|\varphi_\eta\|_{L^p} \approx \|\psi_\eta\|_{L^p} \approx 2^{jn(1/2-1/p)}. \tag{3.12}$$

Moreover, there exist constants $c_1^\diamond, c_2^\diamond > 0$ such that

$$\|\varphi_\eta\|_{L^\infty(B_\eta(c_1^\diamond 2^{-j}))},\ \|\psi_\eta\|_{L^\infty(B_\eta(c_1^\diamond 2^{-j}))} \ge c_2^\diamond 2^{jn/2}. \tag{3.13}$$

See the proof in the appendix.

The following proposition provides a discrete decomposition of $\mathcal{S}'$ and $L^p(\mathbb{S}^n)$ via needlets.

**Proposition 3.2.** $(a)$ *If $f\in\mathcal{S}'$, then*

$$f = \sum_{j=0}^\infty \Psi_j * \overline{\Phi}_j * f \quad \textit{in } \mathcal{S}' \tag{3.14}$$

*and*

$$f = \sum_{\eta\in\mathcal{X}} \langle f, \varphi_\eta\rangle \psi_\eta \quad \textit{in } \mathcal{S}'. \tag{3.15}$$

$(b)$ *If $f\in L^p(\mathbb{S}^n)$, $1\le p\le\infty$, then* $(3.14)-(3.15)$ *hold in $L^p$. Moreover, if $1<p<\infty$, then the convergence in* $(3.14)-(3.15)$ *is unconditional.*

**Proof.** (a) By the definition of $\Phi_j$ and $\Psi_j$ it follows that $\Psi_0 * \overline{\Phi}_0 = \mathsf{P}_0$ and

$$\Psi_j * \overline{\Phi}_j(\omega\cdot\xi) = \sum_{\nu=0}^\infty \overline{\widehat{a}\Big(\frac{\nu}{2^{j-1}}\Big)}\widehat{b}\Big(\frac{\nu}{2^{j-1}}\Big)\mathsf{P}_\nu(\omega\cdot\xi), \quad j\ge 1.$$

Now, as in the proof of Lemma 2.11, (3.4) yields (3.14).

To establish (3.15), we note that $\Psi_j(\xi\cdot\eta)\overline{\Phi_j(\omega\cdot\eta)}$ is a polynomial of degree $< 2^{j+1}$ in $\eta$ and applying the cubature formula from Corollary 2.9, we obtain

$$\begin{aligned}\Psi_j * \overline{\Phi}_j(\omega\cdot\xi) &= \int_{\mathbb{S}^n} \Psi_j(\xi\cdot\eta)\overline{\Phi_j(\omega\cdot\eta)}\, d\mu(\eta)\\ &= \sum_{\eta\in\mathcal{X}_j} c_\eta \Psi_j(\xi\cdot\eta)\overline{\Phi_j(\omega\cdot\eta)} = \sum_{\eta\in\mathcal{X}_j}\psi_\eta(\xi)\overline{\varphi_\eta(\omega)}.\end{aligned}$$

Consequently,

$$\Psi_j * \overline{\Phi}_j * f = \sum_{\eta \in \mathcal{X}_j} \langle f, \varphi_\eta \rangle \psi_\eta,$$

which along with (3.14) implies (3.15).

(b) The proof of (3.14) in $L^p$ is similar to the proof of (2.32) in $L^p$. Then (3.15) in $L^p$ follows as above. The unconditional convergence in $L^p$, $1 < p < \infty$, follows by Proposition 4.3 and Theorem 4.5 below. □

**Remark 3.3.** *Suppose that in the above construction* $\widehat{b} = \widehat{a}$ *and* $\widehat{a} \geq 0$. *So,* $\widehat{a}$ *is as in Lemma 3.1. Then* $\Phi_j = \Psi_j$ *and* $\varphi_\eta = \psi_\eta$. *Now* (3.15) *becomes* $f = \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \psi_\eta$. *It is easily seen* [10] *that this representation holds in* $L^2$ *and*

$$\|f\|_{L^2} = \Big( \sum_{\eta \in \mathcal{X}} |\langle f, \psi_\eta \rangle|^2 \Big)^{1/2}, \quad f \in L^2. \tag{3.16}$$

*This shows that* $\{\psi_\eta : \eta \in \mathcal{X}\}$ *is a tight frame for* $L^2(\mathbb{S}^n)$. *For more details, see* [10].

## 4. Triebel-Lizorkin spaces on $\mathbb{S}^n$

In analogy to the classical case on $\mathbb{R}^n$ (see [4, 5, 18, 19]) the Triebel-Lizorkin spaces on $\mathbb{S}^n$ can be introduced by using Littlewood-Paley decompositions via the kernels $\Phi_j$ defined in (2.26). We assume that $\widehat{a}$ satisfies (2.27)-(2.28).

**Definition 4.1.** *The Triebel-Lizorkin space* $F_p^{\alpha q} := F_p^{\alpha q}(\mathbb{S}^n)$, *where* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, *is defined as the set of all* $f \in \mathcal{S}'$ *such that*

$$\|f\|_{F_p^{\alpha q}} := \Big\| \Big( \sum_{j=0}^{\infty} (2^{\alpha j} |\Phi_j * f(\cdot)|)^q \Big)^{1/q} \Big\|_{L^p} < \infty. \tag{4.1}$$

*Here the* $\ell^q$*-norm is replaced by the sup norm when* $q = \infty$.

**Remark.** As will be shown in Theorem 4.5, the above definition of Triebel-Lizorkin spaces is independent of the specific selection of $\widehat{a}$ satisfying (2.27)-(2.28) in the definition of $\Phi_j$ in (2.26).

**Proposition 4.2.** *The Triebel-Lizorkin space* $F_p^{\alpha q}$ *is a quasi-Banach space which is continuously embedded in* $\mathcal{S}'$ $(F_p^{\alpha q} \hookrightarrow \mathcal{S}')$.

**Proof.** We will only prove that $F_p^{\alpha q} \hookrightarrow \mathcal{S}'$. Then the completeness of $F_p^{\alpha q}$ follows by a standard argument using in addition Fatou's lemma and (2.32).

We only prove the embedding $F_p^{\alpha q} \hookrightarrow \mathcal{S}'$ whenever the functions $\{\Phi_j\}$ in Definition 4.1 are defined by a function $\widehat{a}$ which satisfies (2.29), in addition to (2.27)-(2.28). In Theorem 4.5 below it will be shown that the definition of $F_p^{\alpha q}$ is independent of the specific selection of $\widehat{a}$.

Let $f \in F_p^{\alpha q}$. By Lemma 2.10, $\Phi_j * f \in \Pi_{2^j}(\mathbb{S}^n) \ominus \Pi_{2^{j-2}}(\mathbb{S}^n)$ and hence for $\phi \in \mathcal{S}$

$$\begin{aligned} |\langle \Phi_j * f, \phi \rangle| &= \Big| \int_{\mathbb{S}^n} (\Phi_j * f)(\xi) \sum_{\nu=2^{j-2}}^{2^j} (\mathsf{P}_\nu * \phi)(\xi) d\mu(\xi) \Big| \\ &\leq c 2^{nj/p} \|\Phi_j * f\|_{L^p} \sum_{\nu=2^{j-2}}^{2^j} \|\mathsf{P}_\nu * \phi\|_{L^2} \leq c 2^{-j} \|f\|_{F_p^{\alpha q}} P_r^{**}(\phi), \end{aligned}$$

if $r \geq n/p - s + 1$. Here $P_r^{**}(\phi)$ is the norm from (2.25) and we used Proposition 2.5. From the above it follows that

$$|\langle f, \phi\rangle| \leq \sum_{j=0}^{\infty} |\langle \Phi_j * f, \phi\rangle| \leq c\|f\|_{F_p^{\alpha q}} P_r^{**}(\phi),$$

which gives the desired embedding. $\Box$

We next show that the Triebel-Lizorkin spaces on $\mathbb{S}^n$ can be viewed as a generalization of *potential spaces* (*generalized Sobolev spaces*) *on* $\mathbb{S}^n$, in particular, the $L^p(\mathbb{S}^n)$ spaces, $1 < p < \infty$.

The potential space $H_\alpha^p := H_\alpha^p(\mathbb{S}^n)$, $\alpha > 0$, $1 \leq p \leq \infty$, is defined as the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H_\alpha^p} := \Big\| \sum_{\nu=0}^{\infty} (\nu+1)^\alpha \mathsf{P}_\nu * f \Big\|_{L^p} < \infty, \tag{4.2}$$

where $\mathsf{P}_\nu$ is from (2.1).

**Proposition 4.3.** *We have the following identification:*

$$F_p^{\alpha 2} \sim H_\alpha^p, \quad \alpha \in \mathbb{R},\ 1 < p < \infty, \tag{4.3}$$

*with equivalent norms, and in particular,*

$$F_p^{02} \sim H_0^p \sim L^p, \quad 1 < p < \infty. \tag{4.4}$$

We give the short proof of (4.3) in the appendix.

The following identification of the Hardy spaces $H^p(\mathbb{S}^n)$, $0 < p \leq 1$, on the sphere can be proved in a standard way:

$$F_p^{02} \sim H^p, \quad 0 < p \leq 1, \tag{4.5}$$

*with equivalent norms.*

The proof of (4.5), however, is much longer and will be omitted. It uses atomic and molecular decompositions of Hardy spaces on $\mathbb{S}^n$ (see [2]) and the boundedness of Calderón-Zygmund operators. It follows along the lines of the proof of the corresponding theorem for wavelets in [7].

**Needlet decomposition of Triebel-Lizorkin spaces.** Let $\{\mathcal{X}_j\}_{j=0}^\infty$ be a fixed sequence of sets of almost uniformly $\varepsilon_j$-distributed points on $\mathbb{S}^n$ ($\mathcal{X}_j := \mathcal{X}_{\varepsilon_j}$) with $\varepsilon_j := c^\diamond 2^{-j-2}$ as in Corollary 2.9.

**Definition 4.4.** *The Triebel-Lizorkin sequence space* $\mathbf{f}_p^{\alpha q}$ *is defined as the set of all sequences of complex numbers* $s = \{s_\eta\}_{\eta\in\mathcal{X}}$ *such that*

$$\|s\|_{\mathbf{f}_p^{\alpha q}} := \Big\| \Big( \sum_{\eta\in\mathcal{X}} \Big[ |G_\eta|^{-\alpha/n-1/2} |s_\eta| \mathbb{1}_{G_\eta}(\cdot) \Big]^q \Big)^{1/q} \Big\|_{L^p} < \infty. \tag{4.6}$$

*Here the* $G_\eta$*'s are the spherical caps introduced in* (2.16).

Assuming that $\{\varphi_\eta\}$ and $\{\psi_\eta\}$ are two sequences of analysis and synthesis needlets associated with $\{\mathcal{X}_j\}_{j=0}^\infty$ (see (3.7)-(3.9)), we introduce the operators:

*Analysis operator:* $S_\varphi : f \to \{\langle f, \varphi_\eta\rangle\}_{\eta\in\mathcal{X}}$,

*Synthesis operator:* $T_\psi : \{s_\eta\}_{\eta\in\mathcal{X}} \to \sum_{\eta\in\mathcal{X}} s_\eta \psi_\eta$.

We next give our main result about Triebel-Lizorkin spaces on the sphere. It is an analogue of the fundamental result of Frazier and Jawerth from [4].

**Theorem 4.5.** *If $\alpha \in \mathbb{R}$ and $0 < p < \infty$, $0 < q \le \infty$, then the operators $S_\varphi : F_p^{\alpha q} \to \mathbf{f}_p^{\alpha q}$ and $T_\psi : \mathbf{f}_p^{\alpha q} \to F_p^{\alpha q}$ are bounded and $T_\varphi \circ S_\varphi = \mathrm{Id}$. Consequently, assuming that $f \in \mathcal{S}'$, we have $f \in F_p^{\alpha q}$ if and only if $\{\langle f, \varphi_\eta\rangle\} \in \mathbf{f}_p^{\alpha q}$ and*

$$\|f\|_{F_p^{\alpha q}} \approx \|\{\langle f, \varphi_\eta\rangle\}\|_{\mathbf{f}_p^{\alpha q}} \approx \Big\| \Big( \sum_{j=0}^{\infty} 2^{\alpha j q} \sum_{\eta \in \mathcal{X}_j} |\langle f, \varphi_\eta\rangle \psi_\eta(\cdot)|^q \Big)^{1/q} \Big\|_{L^p}. \tag{4.7}$$

*Also, the definition of $F_p^{\alpha q}$ is independent of the specific selection of $\widehat{a}$ satisfying* (2.27) – (2.28).

For the proof of this theorem we need several lemmas whose proofs are given in the appendix.

**Definition.** For any collection of complex numbers numbers $\{s_\eta\}_{\eta \in \mathcal{X}_j}$, we define

$$s_\eta^* := \sum_{\sigma \in \mathcal{X}_j} \frac{|s_\sigma|}{(1 + 2^j d(\sigma, \eta))^k}, \quad \eta \in \mathcal{X}_j, \tag{4.8}$$

where $k > 0$ is sufficiently large and will be determined later on.

**Lemma 4.6.** *Suppose $g \in \Pi_{2^j}(\mathbb{S}^n)$ and let $a_\eta := \sup_{\xi \in G_\eta} |g(\xi)|$ and*

$$b_\eta := \max\{\inf_{\xi \in G_\omega} |g(\xi)| : \omega \in \mathcal{X}_{j+r}, G_\omega \cap G_\eta \neq \emptyset\}, \quad \eta \in \mathcal{X}_j.$$

*Then there exists $r \ge 1$, depending only on $k$ and $n$, such that*

$$a_\eta^* \approx b_\eta^*, \quad \eta \in \mathcal{X}_j, \tag{4.9}$$

*with constants of equivalence independent of $g$, $j$, and $\eta$.*

**Lemma 4.7.** *Suppose $s > 0$ and $k > n \max\{1, 1/s\}$. Let $\{b_\omega\}_{\omega \in \mathcal{X}_j}$, $j \ge 0$, be a set of complex numbers. Then for $\eta \in \mathcal{X}_j$*

$$b_\eta^* \mathbb{1}_{G_\eta}(\xi) \le c \mathcal{M}_s \Big( \sum_{\omega \in \mathcal{X}_j} |b_\omega| \mathbb{1}_{G_\omega} \Big)(\xi), \quad \xi \in \mathbb{S}^n, \tag{4.10}$$

*with $c = c(k, s, n)$.*

**Lemma 4.8.** *If $k \ge n + 1$, then for $\xi, \eta \in \mathbb{S}^n$*

$$\int_{\mathbb{S}^n} \frac{1}{(1 + 2^j d(\xi, \sigma))^k (1 + 2^j d(\eta, \sigma))^k} \, d\mu(\sigma) \le \frac{c 2^{-jn}}{(1 + 2^j d(\xi, \eta))^k} \tag{4.11}$$

*and*

$$\sum_{\sigma \in \mathcal{X}_j} \frac{1}{(1 + 2^j d(\eta, \sigma))^k (1 + 2^j d(\omega, \sigma))^k} \le \frac{c}{(1 + 2^j d(\eta, \omega))^k} \tag{4.12}$$

*with $c = c(n, k)$.*

**Lemma 4.9.** *If $k \ge n/s$, $s > 0$, and $\eta \in \mathcal{X}_j$, then*

$$\mathcal{M}_s \psi_\eta(\xi) \approx \mathcal{M}_s \Big( |G_\eta|^{-1/2} \mathbb{1}_{G_\eta} \Big)(\xi) \approx \frac{c 2^{jn/2}}{(1 + 2^j d(\xi, \eta))^{n/s}}, \quad \xi \in \mathbb{S}^n, \tag{4.13}$$

*with $c = c(k, s, n)$.*

**Proof of Theorem 4.5.** Suppose $q < \infty$. The proof in the case $q = \infty$ is easier and will be omitted. Fix $0 < s < \min\{p, q\}$ and $k > n \max\{1, 1/s\}$.

We first observe that since $\mathcal{M}_s(\psi_\eta) \approx \mathcal{M}_s(|G_\eta|^{-1/2}\mathbb{1}_{G_\eta})$ (Lemma 4.9), then the right-hand-side equivalence in (4.7) is immediate by applying the maximal inequality (2.19).

Let $\{\Phi_j\}$ be from the definition of Triebel-Lizorkin spaces, defined in (2.26) via a function $\widehat{a}$ satisfying (2.27)-(2.28) (the same as (3.1)-(3.2)). Then by Lemma 3.1, (a) there is a function $\widehat{b}$ satisfying (3.1)-(3.2) such that (3.3) holds true as well. Define $\{\Psi_j\}$ using $\widehat{b}$ as in (3.8) and let $\{\varphi_\eta\}$ and $\{\psi_\eta\}$ be the corresponding needlets defined as in (3.9). Assume also that $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$, $\{\widetilde{\varphi}_\eta\}$, $\{\widetilde{\psi}_\eta\}$ is another needlet system, defined as in (3.7)-(3.9).

We will first prove the boundedness of the operator $T_{\widetilde{\psi}} : \mathbf{f}_p^{\alpha q} \to F_p^{\alpha q}$, where

$$T_{\widetilde{\psi}} s := \sum_{\eta \in \mathcal{X}} s_\eta \widetilde{\psi}_\eta.$$

Let $s = \{s_\eta\}_{\eta \in \mathcal{X}}$ be a finitely supported sequence and set $f := T_{\widetilde{\psi}} s$. The semi-orthogonality of the needlets yields

$$\Phi_j * f = \sum_{\mu=j-1}^{j+1} \sum_{\omega \in \mathcal{X}_\mu} s_\omega \Phi_j * \widetilde{\psi}_\eta.$$

Then for $\eta \in \mathcal{X}_{j-1} \cup \mathcal{X}_j \cup \mathcal{X}_{j+1}$, we have, using (3.10) and (4.11),

$$\begin{aligned} |\Phi_j * \widetilde{\psi}_\eta(\xi)| &= \sqrt{c_\eta}\Big| \int_{\mathbb{S}^n} \widetilde{\Psi}_\mu(\eta \cdot \sigma) \Phi_j(\xi \cdot \sigma)\, d\mu(\sigma) \Big| \\ &\le c 2^{3jn/2} \int_{\mathbb{S}^n} \frac{1}{(1 + 2^j d(\eta, \sigma))^k (1 + 2^j d(\xi, \sigma))^k}\, d\mu(\sigma) \qquad (4.14) \\ &\le \frac{c 2^{jn/2}}{(1 + 2^j d(\xi, \eta))^k}. \end{aligned}$$

Let $\mathcal{X}(\eta) := \{\sigma \in \mathcal{X}_{j-1} \cup \mathcal{X}_j \cup \mathcal{X}_{j+1} : G_\sigma \cap G_\eta \neq \emptyset\}$, where $\mathcal{X}_{-1} := \emptyset$. From above we obtain, for $\xi \in G_\eta$,

$$\begin{aligned} |\Phi_j * f(\xi)| &\le c 2^{jn/2} \sum_{\mu=j-1}^{j+1} \sum_{\omega \in \mathcal{X}_\mu} \frac{|s_\omega|}{(1 + 2^\mu d(\xi, \omega))^k} \\ &\le c 2^{jn/2} \sum_{\mu=j-1}^{j+1} \sum_{\sigma \in \mathcal{X}(\eta) \cap \mathcal{X}_\mu} \sum_{\omega \in \mathcal{X}_\mu} \frac{|s_\omega|}{(1 + 2^\mu d(\sigma, \omega))^k} \\ &\le c \sum_{\sigma \in \mathcal{X}(\eta)} |G_\sigma|^{-1/2} s_\sigma^*, \qquad (|G_\sigma| \approx 2^{-jn}), \end{aligned}$$

where $s_\sigma^*$ is defined in (4.8). Hence

$$\|f\|_{F_p^{\alpha q}} \le c \Big\| \Big( \sum_{\eta \in \mathcal{X}} \Big[ |G_\eta|^{-\alpha/n - 1/2} |s_\eta^*| \mathbb{1}_{G_\eta}(\cdot) \Big]^q \Big)^{1/q} \Big\|_{L^p} = c \|\{s_\eta^*\}\|_{\mathbf{f}_p^{\alpha q}}.$$

Applying now Lemma 4.7 and the maximal inequality (2.19), we obtain

$$\|f\|_{F_p^{\alpha q}} \le c \Big\| \Big( \sum_{\eta \in \mathcal{X}} \mathcal{M}_s \Big( |G_\eta|^{-\alpha/n - 1/2} |s_\eta| \mathbb{1}_{G_\eta}(\cdot) \Big)^q \Big)^{1/q} \Big\|_{L^p} \le c \|\{s_\eta\}\|_{\mathbf{f}_p^{\alpha q}}. \qquad (4.15)$$

We now turn to an arbitrary sequence $s \in \mathbf{f}_p^{\alpha q}$. Estimate (4.15) holds with an arbitrary $\widehat{a}$ (in the definition of $\{\Phi_j\}$) satisfying (2.27)-(2.28). So, assume for an instant that $\widehat{a}$ satisfies (2.29) as well. Then we can use Proposition 4.2 which was proved with such $\widehat{a}$'s. Therefore, by (4.15), Proposition 4.2, and the fact that finitely supported sequence are dense in $\mathbf{f}_p^{\alpha q}$ it follows that $T_{\widetilde{\psi}} s := \sum_{\eta\in\mathcal{X}} s_\eta \widetilde{\psi}_\eta$ is well defined in $\mathcal{S}'$. Finally, by a limiting argument it follows that (4.15) holds for all sequences $s \in \mathbf{f}_p^{\alpha q}$. Thus the operator $T_{\widetilde{\psi}} : \mathbf{f}_p^{\alpha q} \to F_p^{\alpha q}$ is bounded.

We next prove the boundedness of the operator $S_\varphi : F_p^{\alpha q} \to \mathbf{f}_p^{\alpha q}$. Let $f \in F_p^{\alpha q}$. For $\eta \in \mathcal{X}_j$, let

$$A_\eta := \sup_{\xi\in G_\eta} |\overline{\Phi}_j * f(\xi)| \text{ and } B_\eta := \max\{\inf_{\xi\in G_\omega} |\overline{\Phi}_j * f(\xi)| : \omega \in \mathcal{X}_{j+r}, G_\omega \cup G_\eta \neq \emptyset\}.$$

By Lemma 2.10, $\overline{\Phi}_j * f \in \Pi_{2^j}(\mathbb{S}^n)$ and applying Lemma 4.6, we can select $r \geq 1$ ($r = r(k,n)$) so that $A_\eta^* \leq cB_\eta^*$, $\eta \in \mathcal{X}_j$. Then

$$\begin{aligned} |\langle f, \varphi_\eta\rangle| &\leq c|G_\eta|^{1/2}|\overline{\Phi}_j * f(\eta)| \leq c|G_\eta|^{1/2}A_\eta \\ &\leq c|G_\eta|^{1/2}A_\eta^* \leq c|G_\eta|^{1/2}B_\eta^* \end{aligned}$$

and hence

$$\begin{aligned} \|\{\langle f, \varphi_\eta\rangle\}\|_{\mathbf{f}_p^{\alpha q}} &\leq c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}\Big(\sum_{\eta\in\mathcal{X}_j} A_\eta \mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p} \\ &\leq c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}\Big(\sum_{\eta\in\mathcal{X}_j} B_\eta^* \mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p} \\ &\leq c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}\mathcal{M}_s\Big(\sum_{\eta\in\mathcal{X}_j} B_\eta \mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p} \\ &\leq c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}\Big(\sum_{\eta\in\mathcal{X}_j} B_\eta \mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p}, \end{aligned}$$

where we used Lemma 4.7 and the maximal inequality (2.19).

Let

$$m_\sigma := \inf_{\xi\in G_\sigma} |\overline{\Phi}_j * f(\xi)|, \quad \sigma \in \mathcal{X}_{j+r}.$$

For $\eta \in \mathcal{X}_j$, we denote $\mathcal{X}_{j+r}(\eta) := \{\omega \in \mathcal{X}_{j+r} : G_\omega \cap G_\eta \neq \emptyset\}$. Note that $\#\mathcal{X}_{j+r}(\eta) \leq c(r,n)$. Since $r$ depends only on $k$ and $n$, then for $\eta \in \mathcal{X}_j$ and $\omega \in \mathcal{X}_{j+r}(\eta)$, we have

$$B_\eta = \max_{\lambda\in\mathcal{X}_{j+r}(\eta)} m_\lambda \leq c\sum_{\sigma\in\mathcal{X}_{j+r}} \frac{m_\sigma}{(1+2^{j+r}d(\omega,\sigma))^k} = cm_\omega^*, \quad c = c(r,n),$$

and hence

$$B_\eta \mathbb{1}_{G_\eta} \leq c\sum_{\omega\in\mathcal{X}_{j+r}(\eta)} m_\omega^* \mathbb{1}_{G_\omega}, \quad \xi \in \mathbb{S}^n.$$

We use the above and again Lemma 4.7 and the maximal inequality (2.19) to obtain

$$\begin{aligned}
\|\{\langle f,\varphi_\eta\rangle\}\|_{\mathbf{f}_p^{\alpha q}} &\le c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}\Big(\sum_{\eta\in\mathcal{X}_{j+r}} m_\eta^*\mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
&\le c\Big\|\Big(\sum_{j=0}^\infty \mathcal{M}_s\Big(2^{\alpha j}\sum_{\eta\in\mathcal{X}_{j+r}} m_\eta\mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
&\le c\Big\|\Big(\sum_{j=0}^\infty \Big(2^{\alpha j}\sum_{\eta\in\mathcal{X}_{j+r}} m_\eta\mathbb{1}_{G_\eta}\Big)^q\Big)^{1/q}\Big\|_{L^p}\\
&\le c\Big\|\Big(\sum_{j=0}^\infty 2^{\alpha jq}|\overline{\Phi}_j * f|^q\Big)^{1/q}\Big\|_{L^p} = c\|f\|_{F_p^{\alpha q}},
\end{aligned}$$

where we also used that

$$\sum_{\eta\in\mathcal{X}_{j+r}} m_\eta\mathbb{1}_{G_\eta}(\xi) \le c|\overline{\Phi}_j * f(\xi)|, \quad \xi\in\mathcal{X}_j.$$

Hence the operator $S_\varphi : F_p^{\alpha q}\to\mathbf{f}_p^{\alpha q}$ is bounded.

The identity $T_\psi\circ S_\varphi = Id$ is immediate from Theorem 3.2.

It remains to show that the definition of the Triebel-Lizorkin spaces is independent of the specific selection of $\widehat{a}$ satisfying (2.27)-(2.28). Assume that $\{\Phi_j\}$ and $\{\widetilde{\Phi}_j\}$ are two sequences of functions defined as in (3.8) by two different functions $\widehat{a}$ satisfying (2.27)-(2.28). Using Lemma 3.1 as above, there exist two associated needlet systems, say, $\{\Phi_j\}$, $\{\Psi_j\}$, $\{\varphi_\eta\}$, $\{\psi_\eta\}$ and $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$, $\{\widetilde{\varphi}_\eta\}$, $\{\widetilde{\psi}_\eta\}$. Let us denote for a moment by $\|f\|_{F_p^{\alpha q}(\Phi)}$ and $\|f\|_{F_p^{\alpha q}(\widetilde{\Phi})}$ the $F$-norms defined by using $\{\Phi_j\}$ and $\{\widetilde{\Phi}_j\}$, respectively. Then by the above proof it follows that

$$\|f\|_{F_p^{\alpha q}(\Phi)} \le c\|\{\langle f,\widetilde{\varphi}_\eta\rangle\}\|_{\mathbf{f}_p^{\alpha q}} \le c\|f\|_{F_p^{\alpha q}(\overline{\widetilde{\Phi}})}.$$

Consequently, the definition of $F_p^{\alpha q}$ is independent of the specific selection of $\widehat{a}$ satisfying (2.27)-(2.28) in the definition of the functions $\{\Phi_j\}$. □

## 5. Besov spaces on $\mathbb{S}^n$

In our treatment of Besov spaces on the sphere, we will use the approach of Frazier and Jawerth [3] (see also [5]). We refer to [11, 18] as general references for Besov spaces.

**Definition 5.1.** *The Besov space $B_p^{\alpha q} := B_p^{\alpha q}(\mathbb{S}^n)$, where $\alpha\in\mathbb{R}$, $0<p,q\le\infty$, is defined as the set of all $f\in\mathcal{S}'$ such that*

$$\|f\|_{B_p^{\alpha q}} := \Big(\sum_{j=0}^\infty \Big(2^{\alpha j}\|\Phi_j * f\|_{L^p}\Big)^q\Big)^{1/q} < \infty, \tag{5.1}$$

*where the $\ell^q$-norm is replaced by the sup-norm if $q=\infty$. Here the kernels $\{\Phi_j\}$ are defined in* (2.26) *with $\widehat{a}$ satisfying* (2.27) − (2.28).

It follows by Theorem 5.5 that the above definition of Besov spaces is independent of the specific selection of $\widehat{a}$. Further, the Besov space $B_p^{\alpha q}$ is a quasi-Banach space which is continuously embedded in $\mathcal{S}'$. The proof of this is similar to the one for Triebel-Lizorkin spaces and will be omitted.

We need the following embedding result.

**Proposition 5.2.** *If $\alpha > 0$ and $0 < p, q \le \infty$, then $B_p^{\alpha q}$ is continuously embedded in $L^p$, i.e. each $f \in B_p^{\alpha q}$ can be identified as a function in $L^p$ and*

$$\|f\|_{L^p} \le c\|f\|_{B_p^{\alpha q}}. \tag{5.2}$$

The proof of this proposition is standard and easy and will be omitted.

**Characterization of Besov spaces via polynomial approximation.** We now want to make the connection between our treatment of Besov on the sphere and $L^p$-polynomial approximation on the sphere. Recall that $E_m(f)_p$ denotes the best approximation of $f \in L^p$ from $\Pi_m(\mathbb{S}^n)$ (see (2.8)).

**Proposition 5.3.** *If $\alpha > 0$, $1 \le p \le \infty$, and $0 < q \le \infty$, then $f \in B_p^{\alpha q}$ if and only if*

$$\|f\|^A_{B_p^{\alpha q}} := \|f\|_{L^p} + \Big(\sum_{j=0}^{\infty} (2^{\alpha j} E_{2^j}(f)_p)^q\Big)^{1/q} < \infty. \tag{5.3}$$

*Moreover,*

$$\|f\|^A_{B_p^{\alpha q}} \approx \|f\|_{B_p^{\alpha q}}. \tag{5.4}$$

**Proof.** Suppose that the polynomials $\{\Phi_j\}$ are defined by (2.26) with $\widehat{a}$ satisfying (2.27)-(2.29). Let $f \in B_p^{\alpha q}$. Then $f \in L^p$ (Theorem 5.2) and by Lemma 2.11 $f = \sum_{j=0}^\infty \Phi_j * f$ in $L^p$. Since $\Phi_j * f \in \Pi_{2^j}$, we have

$$E_{2^m}(f)_p \le \sum_{j=m+1}^{\infty} \|\Phi_j * f\|_{L^p}, \quad m \ge 0. \tag{5.5}$$

A standard argument employing (5.5) and Theorem 5.2 leads to the estimate $\|f\|^A_{B_p^{\alpha q}} \le c\|f\|_{B_p^{\alpha q}}$.

In the other direction it is simpler. For $g \in \Pi_{2^{j-2}}$ ($j \ge 2$), we have using Lemma 2.4, $\Phi_j * f = \Phi_j * (f - g)$ and hence again by the same lemma, $\|\Phi_j * f\|_{L^p} \le c\|f - g\|_{L^p}$. Consequently,

$$\|\Phi_j * f\|_{L^p} \le cE_{2^{j-2}}(f)_p, \quad j \ge 2, \quad \text{and} \quad \|\Phi_j * f\|_{L^p} \le c\|f\|_{L^p}.$$

From this, we obtain at once $\|f\|_{B_p^{\alpha q}} \le c\|f\|^A_{B_p^{\alpha q}}$. □

**Needlet decomposition of Besov spaces.** We again fix a sequence $\{\mathcal{X}_j\}_{j=0}^\infty$ of sets of almost uniformly $\varepsilon_j$-distributed points on $\mathbb{S}^n$ ($\mathcal{X}_j := \mathcal{X}_{\varepsilon_j}$) with $\varepsilon_j := c^\diamond 2^{-j-2}$ as in Corollary 2.9.

**Definition 5.4.** *The Besov sequence space $\mathbf{b}_p^{\alpha q}$, where $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, is defined as the set of all sequences of complex numbers $s = \{s_\eta\}_{\eta \in \mathcal{X}}$ such that*

$$\|s\|_{\mathbf{b}_p^{\alpha q}} := \Big(\sum_{j=0}^{\infty} \Big[2^{j(\alpha+n/2-n/p)}\Big(\sum_{\eta\in\mathcal{X}_j} |s_\eta|^p\Big)^{1/p}\Big]^q\Big)^{1/q} < \infty \tag{5.6}$$

*with obvious modifications when $p = \infty$ or $q = \infty$.*

In the next theorem, we assume that $\{\Phi_j\}$, $\{\Psi_j\}$, $\{\varphi_\eta\}$, $\{\psi_\eta\}$ is a needlet system (see (3.7)-(3.9)). We also recall the *analysis operator:* $S_\varphi : f \to \{\langle f, \varphi_\eta\rangle\}_{\eta\in\mathcal{X}}$, and the *synthesis operator:* $T_\psi : \{s_\eta\}_{\eta\in\mathcal{X}} \to \sum_{\eta\in\mathcal{X}} s_\eta\psi_\eta$.

**Theorem 5.5.** *If $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, then the operators $S_\varphi : B_p^{\alpha q} \to \mathbf{b}_p^{\alpha q}$ and $T_\psi : \mathbf{b}_p^{\alpha q} \to B_p^{\alpha q}$ are bounded and $T_\varphi \circ S_\varphi = \mathrm{Id}$. Consequently, assuming that $f \in \mathcal{S}'$, we have $f \in B_p^{\alpha q}$ if and only if $\{\langle f, \varphi_\eta\rangle\} \in \mathbf{b}_p^{\alpha q}$ and*

$$\|f\|_{B_p^{\alpha q}} \approx \|\{\langle f, \varphi_\eta\rangle\}\|_{\mathbf{b}_p^{\alpha q}} \approx \Big(\sum_{j=0}^{\infty}\Big[2^{\alpha j}\Big(\sum_{\eta\in\mathcal{X}_j}\|\langle f, \varphi_\eta\rangle\psi_\eta\|_{L^p}^p\Big)^{1/p}\Big]^q\Big)^{1/q}. \tag{5.7}$$

*Furthermore, the definition of $B_p^{\alpha q}$ is independent of the choice of $\widehat{a}$ satisfying* (2.27) – (2.28).

To prove Theorem 5.5, we need two additional lemmas.

**Lemma 5.6.** *If $\{s_\eta\}_{\eta\in\mathcal{X}_j}$ is a set of numbers ($j \ge 0$), $0 < p \le \infty$, and $k > n\max\{1/p, 1\}$, then*

$$\Big\|\sum_{\eta\in\mathcal{X}_j}\frac{|s_\eta|}{(1+2^j d(\cdot,\eta))^k}\Big\|_{L^p} \le c2^{-jn/p}\Big(\sum_{\eta\in\mathcal{X}_j}|s_\eta|^p\Big)^{1/p}, \tag{5.8}$$

*with $c = c(n, k, p)$.*

This lemma is an immediate consequence of Lemma 4.7 and the maximal inequality (2.19).

**Lemma 5.7.** *If $g \in \Pi_{2^j}(\mathbb{S}^n)$ and $0 < p \le \infty$, then*

$$\Big(\sum_{\eta\in\mathcal{X}_j}\sup_{\xi\in G_\eta}|g(\xi)|^p\Big)^{1/p} \le c2^{jn/p}\|g\|_{L^p}, \tag{5.9}$$

*where the $G_\eta$'s are defined in* (2.16) *and $c = c(p, n)$.*

The proof of this lemma is given in the appendix.

**Proof of Theorem 5.5.** We first note that the right-hand-side equivalence in (5.7) follows immediately by (3.12).

We proceed further similarly as in the proof of Theorem 4.5. Suppose $p, q < \infty$. In the other cases the proof is similar. Let $\{\Phi_j\}$ be defined by (2.26) via a function $\widehat{a}$ satisfying (2.27)-(2.28). Then by Lemma 3.1 there is a function $\widehat{b}$ satisfying (2.27)-(2.28) such that (2.29) holds true as well. Define $\{\Psi_j\}$ using $\widehat{b}$ as in (2.26) and let $\{\varphi_\eta\}$ and $\{\psi_\eta\}$ be the corresponding needlets defined as in (3.9). Assume also that $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$, $\{\widetilde{\varphi}_\eta\}$, $\{\widetilde{\psi}_\eta\}$ is another needlet system, defined as in (3.7)-(3.9).

We will first prove the boundedness of the operator $T_{\widetilde{\psi}} : \mathbf{b}_p^{\alpha q} \to B_p^{\alpha q}$, where

$$T_{\widetilde{\psi}} s := \sum_{\eta\in\mathcal{X}} s_\eta\widetilde{\psi}_\eta.$$

Let $s = \{s_\eta\}_{\eta\in\mathcal{X}}$ be finitely supported and denote $f := T_{\widetilde{\psi}} s$. By the semi-orthogonality of the needlets

$$\Phi_j * f = \sum_{\mu=j-1}^{j+1}\sum_{\eta\in\mathcal{X}_\mu} s_\eta \Phi_j * \widetilde{\psi}_\eta, \quad \mathcal{X}_{-1} := \emptyset.$$

Then for $\eta \in \mathcal{X}_{j-1} \cup \mathcal{X}_j \cup \mathcal{X}_{j+1}$, we have exactly as in the proof Theorem 4.5 (see (4.14))

$$|\Phi_j * \widetilde{\psi}_\eta(\xi)| \le \frac{c2^{jn/2}}{(1+2^j d(\xi,\eta))^k}.$$

Applying Lemma 5.6, we infer

$$\begin{aligned}\|\Phi_j * f\|_{L^p} &\leq c2^{jn/2}\Big\| \sum_{\mu=j-1}^{j+1} \sum_{\eta\in\mathcal{X}_\mu} \frac{|s_\eta|}{(1+2^j d(\cdot,\eta))^k}\Big\|_{L^p} \\ &\leq c2^{j(n/2-n/p)} \sum_{\mu=j-1}^{j+1} \Big(\sum_{\eta\in\mathcal{X}_\mu} |s_\eta|^p\Big)^{1/p}.\end{aligned}$$

Substituting this estimate in the definition of $\|\cdot\|_{B_p^{\alpha q}}$, we obtain

$$\|f\|_{B_p^{\alpha q}} \leq c\|\{s_\eta\}\|_{\mathbf{b}_p^{\alpha q}}. \tag{5.10}$$

Finally, as in the proof of Theorem 4.5, we use the continuous embedding $B_p^{\alpha q} \hookrightarrow \mathcal{S}'$ and a limiting argument to conclude that $T_{\widetilde{\psi}} s \in \mathcal{S}'$ and (5.10) hold for an arbitrary sequence $s \in \mathbf{b}_p^{\alpha q}$.

It remains to proof the boundedness of the operator $S_\varphi : B_p^{\alpha q} \to \mathbf{b}_p^{\alpha q}$. Let $f \in B_p^{\alpha q}$. Using the definition of $\Phi_j$ and $\varphi_\eta$, we have

$$\|\{\langle f,\varphi\rangle\}\|_{\mathbf{b}_p^{\alpha q}} \leq c\Big(\sum_{j=0}^{\infty} \Big[2^{j(\alpha-n/p)}\Big(\sum_{\eta\in\mathcal{X}_j} |\overline{\Phi}_j * f(\eta)|^p\Big)^{1/p}\Big]^q\Big)^{1/q}. \tag{5.11}$$

By Lemma 2.10, $\overline{\Phi}_j * f \in \Pi_{2^j}$ and then, using Lemma 5.7,

$$\Big(\sum_{\eta\in\mathcal{X}_j} |\overline{\Phi}_j * f(\eta)|^p\Big)^{1/p} \leq c2^{jn/p}\|\overline{\Phi}_j * f\|_{L^p}.$$

This estimate coupled with (5.11) gives

$$\|\{\langle f,\varphi\rangle\}\|_{\mathbf{b}_p^{\alpha q}} \leq c\|f\|_{B_p^{\alpha q}}. \tag{5.12}$$

Consequently, the operator $S_\varphi : B_p^{\alpha q} \to \mathbf{b}_p^{\alpha q}$ is bounded.

The identity $T_\psi \circ S_\varphi = Id$ follows by Theorem 3.2.

Finally, one repeats the argument from the proof of Theorem 4.5 to show the independence of the definition of Besov spaces of the specific selection of $\widehat{a}$ satisfying (2.27)-(2.28). $\square$

## 6. Application of Besov spaces to nonlinear approximation on $\mathbb{S}^n$

Our goal in this section is the development of nonlinear n-term approximation from needlet systems on $\mathbb{S}^n$.

For simplicity, suppose that $\{\psi_\eta\}_{\eta\in\mathcal{X}}$ is a needlet system with $\varphi_\eta = \psi_\eta$, defined as in (3.7)-(3.9) with $\widehat{b} = \widehat{a}$, $\widehat{a} \geq 0$, and $\widehat{a}$ satisfying (3.5), i.e.

$$\widehat{a}^2(t) + \widehat{a}^2(2t) = 1, \quad t \in [1/2, 1].$$

Hence $\{\psi_\eta\}$ are real-valued.

We let $\Sigma_m$ denote the nonlinear set consisting of all functions $g$ of the form

$$g = \sum_{\eta\in\Lambda} a_\eta \phi_\eta,$$

where $\Lambda \subset \mathcal{X}$, $\#\Lambda \leq m$, and $\Lambda$ is allowed to vary with $g$. We denote by $\sigma_m(f)_p$ the error of best $L^p$-approximation to $f \in L^p(\mathbb{S}^n)$ from $\Sigma_m$ (best $m$-term approximation):

$$\sigma_m(f)_p := \inf_{g\in\Sigma_m} \|f-g\|_p.$$

Here and in the following, we use the abbreviated notation $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{S}^n)}$. The approximation will take place in $L^p$, $0 < p < \infty$.

In the following we will be assuming that $0 < p < \infty$, $\alpha > 0$, and $1/\tau := \alpha/n + 1/p$. Denote briefly $B^\alpha_\tau := B^{\alpha\tau}_\tau$. By Theorem 5.5 and (3.12) it follows that

$$\|f\|_{B^\alpha_\tau} \approx \Big(\sum_{\eta\in\mathcal{X}} \|\langle f, \psi_\eta\rangle\psi_\eta\|_p^\tau\Big)^{1/\tau}. \tag{6.1}$$

The embedding of $B^\alpha_\tau$ into $L^p$ will play a critical role in our development here.

**Proposition 6.1.** *If $f \in B^\alpha_\tau$, then $f$ can be identified as a function $f \in L^p$ and*

$$\|f\|_p \le \Big\|\sum_{\eta\in\mathcal{X}} |\langle f, \psi_\eta\rangle\psi_\eta(\cdot)|\Big\|_p \le c\|f\|_{B^\alpha_\tau}. \tag{6.2}$$

We now state our main result in this section.

**Theorem 6.2. [Jackson estimate]** *If $f \in B^\alpha_\tau$, then*

$$\sigma_m(f)_p \le cm^{-\alpha/n}\|f\|_{B^\alpha_\tau}, \tag{6.3}$$

*where $c$ depends only on $\alpha$, $p$, and the parameters of the needlet system.*

The proofs of Proposition 6.1 and Theorem 6.2 rest on the following lemma.

**Lemma 6.3.** *Let $0 < p < \infty$. Suppose $F := \sum_{\eta\in\mathcal{E}} |a_\eta\psi_\eta|$, where $\mathcal{E} \subset \mathcal{X}$, $\#\mathcal{E} \le m$ and $\|a_\eta\psi_\eta\|_p \le A$ for all $\eta \in \mathcal{E}$. Then*

$$\|F\|_p \le cAm^{1/p}. \tag{6.4}$$

**Proof.** Let $1 < p < \infty$ (the case $p \le 1$ is trivial). Choose $0 < s < \min\{1, p\}$ and $k > n\max\{1, 1/s\}$. By the hypothesis of the lemma, (3.12), and the fact that $|G_\eta| \approx 2^{-jn}$ if $\eta \in \mathcal{X}_j$, it follows that

$$|a_\eta| \le cA2^{jn(1/p-1/2)} \le cA|G_\eta|^{1/2-1/p}, \quad \eta \in \mathcal{X}_j. \tag{6.5}$$

By Lemma 4.9, $|\psi_\eta(\xi)| \le c\mathcal{M}_s(|G_\eta|^{-1/2}\mathbb{1}_{G_\eta})(\xi)$. We use this, (6.5), and the maximal inequality (2.19) to obtain

$$\|F\|_p \le c\Big\|\sum_{\eta\in\mathcal{E}} \mathcal{M}_s(|a_\eta||G_\eta|^{-1/2}\mathbb{1}_{G_\eta})\Big\|_p \le cA\Big\|\sum_{\eta\in\mathcal{E}} |G_\eta|^{-1/p}\mathbb{1}_{G_\eta}\Big\|_p.$$

Denote $G := \bigcup_{\eta\in\mathcal{E}} G_\eta$ and $\mathcal{G}(\xi) := \min\{|G_\eta| : \eta \in \mathcal{E}, \xi \in G_\eta\}$ ($\mathcal{G}(\xi) = 0$ if $\xi \notin G$). Evidently, if $\xi \in G_\omega$ for some $\omega \in \mathcal{E}$, then

$$\sum_{\eta\in\mathcal{E},\, \xi\in G_\eta, |G_\eta|\ge|G_\omega|} (|G_\omega|/|G_\eta|)^{1/p} \le c\sum_{\nu=0}^\infty 2^{-\nu n/p} \le c_1 < \infty.$$

Hence

$$\sum_{\eta\in\mathcal{E}} |G_\eta|^{-1/p}\mathbb{1}_{G_\eta}(\xi) \le c_1\mathcal{G}(\xi)^{-1/p}, \quad \xi \in \mathbb{S}^n.$$

Consequently,

$$\begin{aligned}\|F\|_p &\le cA\|\mathcal{G}(\xi)^{-1/p}\|_p = cA\Big(\int_G \mathcal{G}(\xi)^{-1}d\mu(\xi)\Big)^{1/p}\\ &\le cA\Big(\sum_{\eta\in\mathcal{E}} \frac{1}{|G_\eta|}\int_{\mathbb{S}^n} \mathbb{1}_{G_\eta}d\mu\Big)^{1/p} = cA(\#\mathcal{E})^{1/p} \le cAm^{1/p}.\end{aligned}$$

$\square$

**Proof of Proposition 6.1 and Theorem 6.2.** Denote briefly

$$N(f) := \Big(\sum_{j=0}^{\infty} \|\langle f, \psi_\eta\rangle \psi_\eta\|_p^\tau\Big)^{1/\tau} \quad \text{and} \quad a_\eta := \langle f, \psi_\eta\rangle.$$

Assume $N(f) > 0$. Let $\{a_{\eta_j}\psi_{\eta_j}\}_{j=1}^\infty$ be a rearrangement of the sequence $\{a_\eta \psi_\eta\}_{\eta\in\mathcal{X}}$ so that

$$\|a_{\eta_1}\psi_{\eta_1}\|_p \ge \|a_{\eta_2}\psi_{\eta_2}\|_p \ge \dots.$$

Set $S_m := \sum_{j=1}^m a_{\eta_j}\psi_{\eta_j}$. We will show that

$$\|f - S_m\|_p \le c m^{-\alpha/n} N(f), \quad m \ge 1. \tag{6.6}$$

*Case 1.* $0 < p < 1$. Since $\tau < p$, we have

$$\Big\|\sum_j |a_{\eta_j}\psi_{\eta_j}(\cdot)|\Big\|_p \le \Big(\sum_j \|a_{\eta_j}\psi_{\eta_j}\|_p^p\Big)^{1/p} \le \Big(\sum_j \|a_{\eta_j}\psi_{\eta_j}\|_p^\tau\Big)^{1/\tau} = N(f)$$

which yields Proposition 6.1 in this case.

To estimate $\|f - S_m\|_p$ we will use the following inequality: If $x_1 \ge x_2 \ge \dots \ge 0$ and $0 < \tau < p$, then

$$\Big(\sum_{j=m+1}^\infty x_j^p\Big)^{1/p} \le m^{1/p-1/\tau}\Big(\sum_{j=1}^\infty x_j^\tau\Big)^{1/\tau}. \tag{6.7}$$

For completeness the proof of this simple inequality is given in the appendix. Using Proposition 6.1 and (6.7) with $x_j := \|a_{\eta_j}\psi_{\eta_j}\|_p$, we obtain

$$\begin{aligned} \|f - S_m\|_p &\le \Big\|\sum_{j=m+1}^\infty |a_{\eta_j}\psi_{\eta_j}(\cdot)|\Big\|_p \le \Big(\sum_{j=m+1}^\infty \|a_{\eta_j}\psi_{\eta_j}\|_p^p\Big)^{1/p} \\ &\le m^{1/p-1/\tau}\Big(\sum_{j=1}^\infty \|a_{\eta_j}\psi_{\eta_j}\|_p^\tau\Big)^{1/\tau} = m^{-\alpha/n}N(f), \end{aligned}$$

which proves (6.6) in Case 1.

*Case 2.* $1 \le p < \infty$. We first note that the argument that follows with $m = 0$ ($S_0 = 0$) gives Proposition 6.1 in this case. So, we will use Proposition 6.1 in the proof below.

Denote

$$J_\nu := \{j : 2^{-\nu}N(f) < \|a_{\eta_j}\psi_{\eta_j}\|_p \le 2^{-\nu+1}N(f)\}.$$

Then

$$\bigcup_{\nu\le\mu} J_\nu = \{j : \|a_{\eta_j}\psi_{\eta_j}\|_p > 2^{-\mu}N(f)\}$$

and hence, by the definition of $N(f)$,

$$\sum_{\nu\le\mu} \#J_\nu \le \#\Big(\bigcup_{\nu\le\mu} J_\nu\big) \le 2^{\mu\tau}. \tag{6.8}$$

Consequently,

$$\#J_\mu \le \sum_{\nu\le\mu} \#J_\nu \le 2^{\mu\tau}. \tag{6.9}$$

Let $m \ge 0$ and denote $M := \sum_{\mu\le m} J_\mu$. By (6.8), $M \le 2^{m\tau}$. Let $F_\mu := \sum_{j\in J_\mu} |a_{\eta_j}\psi_{\eta_j}|$. Using Lemma 6.3 and (6.9) we obtain

$$\begin{aligned}\|f - S_M\|_p &\le \|\sum_{\mu=m+1}^{\infty} F_\mu\|_p \le \sum_{\mu=m+1}^{\infty} \|F_\mu\|_p \\ &\le c\sum_{\mu=m+1}^{\infty} 2^{-\mu} N(f)(\#J_\mu)^{1/p} \le cN(f)\sum_{\mu=m+1}^{\infty} 2^{-\mu(1-\tau/p)} \\ &\le cN(f)2^{-m(1-\tau/p)} \le c2^{-m\tau\alpha/n}N(f).\end{aligned}$$

Consequently, $\|f - S_{[2^{m\tau}]}\|_p \le c2^{-m\tau\alpha/n}N(f)$ for $m \ge 0$, which yields (6.6). □

The **grand open problem** here is whether the following Bernstein type estimate holds:

$$\|g\|_{B^\alpha_\tau} \le cm^{\alpha/n}\|g\|_p \quad \text{for } g \in \Sigma_m, \quad 1 < p < \infty. \tag{6.10}$$

The validity of this estimate along with the Jackson estimate from Theorem 6.2 would enable one to obtain a complete characterization of the rates (approximation spaces) of nonlinear $m$-term $L^p$-approximation from the needlet system $\{\psi_\eta\}$ via Besov spaces and interpolation (see e.g. [12]).

Needlets as well as wavelets are not suitable for nonlinear $m$-term approximation in $L^\infty$. Nonlinear approximation in $BMO(\mathbb{S}^n)$ should be considered instead. It is also appropriate to consider nonlinear $m$-term approximation from needlet systems in the Hardy spaces $H^p$ $(0 < p \le 1)$ on the sphere. Jackson estimates for nonlinear $m$-term approximation from needlets in BMO and $H^p$ similar to (6.3) can be proved. We do not present such estimates here since we lack the corresponding Bernstein estimates for a comprehensive theory.

## 7. Appendix

**Proof of (3.12)-(3.13).** By (3.7)-(3.8) it readily follows that

$$\|\Phi_j(\bullet\cdot\eta)\|_{L^2} \approx \|\Psi_j(\bullet\cdot\eta)\|_{L^2} \approx 2^{jn/2}. \tag{7.1}$$

Fix $\eta \in \mathbb{S}^n$ and denote briefly $F(\xi) := \Phi_j(\xi\cdot\eta)$. The estimate $\|F\|_{L^p} \le c2^{jn(1-1/p)}$ is immediate from (3.10).

In the other direction, consider first the case when $0 < p \le 2$. Since $\|F\|_{L^2} \approx 2^{jn/2}$ and $\|F\|_{L^\infty} \le c2^{jn}$ from (3.10), then

$$c2^{jn} \le \|F\|^2_{L^2} \le \|F\|^p_{L^p}\|F\|^{2-p}_{L^\infty} \le 2^{jn(2-p)}\|F\|^p_{L^p}$$

and hence

$$\|F\|_{L^p} \approx 2^{jn(1-1/p)}. \tag{7.2}$$

If $2 < p < \infty$, then using Hölder's inequality and (7.2) with $p < 2$, we have

$$c2^{jn} \le \int_{\mathbb{S}^n} |F|^2\, d\mu \le \|F\|_{L^p}\|F\|_{L_{p'}} \le c\|F\|_{L^p}2^{jn/p}$$

and (7.2) holds again. The other equivalences in (3.12) follow from above and (3.9).

Form (3.10) we have $\|\varphi_\eta\|_{L^\infty} \le c2^{jn/2}$. From this and by (3.10) we obtain, for $0<\rho<\pi$

$$\begin{aligned} 0<c_2 &< \|\varphi_\eta\|_{L^2}^2 \le \|\varphi_\eta\|^2_{L^\infty(B_\eta(\rho))}|B_\eta(\rho)| + \int_{\mathbb{S}^n\setminus B_\eta(\rho)} \frac{c_k^2 2^{jn}}{(1+2^j d(\xi,\eta))^{2k}}d\mu(\xi) \\ &\le c'\rho^n\|\varphi_\eta\|^2_{L^\infty(B_\eta(\rho))} + \frac{c''}{(1+2^j\rho)^{2k-n}}, \quad 2k>n, \end{aligned}$$

where $c', c'' >$ depend only on $n$, $k$, and $c_k$. Choose $\rho = c_1^\diamond 2^{-j}$ so that

$$\frac{c''}{(1+c_1^\diamond)^{2k-n}} < \frac{c_2}{2}$$

Then from above $\|\varphi_\eta\|^2_{L^\infty(B_\eta(c_1^\diamond 2^{-j}))} \ge c2^{jn}$ which yields (3.13). We estimate the $L^\infty$-norm of $\psi$ exactly in the same way. This completes the proof of (3.12)-(3.13). $\square$

**Proof of Proposition 4.3.** For a sequence $\varepsilon = \{\varepsilon_j\}_{j\ge 0}$ with $\varepsilon_j = \pm 1$, we define

$$m_\varepsilon(t) := \sum_{j=1}^\infty \frac{\varepsilon_j 2^{j\alpha}}{(t+1)^\alpha}\widehat{a}\Big(\frac{t}{2^{j-1}}\Big).$$

and we let $\varepsilon_0 := \{1,1,\dots\}$. Evidently,

$$\sum_{j=1}^\infty \varepsilon_j 2^{j\alpha}\Phi_j * f = \sum_{\nu=1}^\infty m_\varepsilon(\nu)(\nu+1)^\alpha \mathsf{P}_\nu * f.$$

It is readily seen that for $r=0,1,\dots,$

$$\|t^r m_\varepsilon^{(r)}(t)\|_{L^\infty} \le c(r) < \infty \text{ and } \|t^r(1/m_{\varepsilon_0})^{(r)}(t)\|_{L^\infty[1,\infty)} \le c(r) < \infty. \tag{7.3}$$

By [16] this yields that $m_\varepsilon$ (for any $\varepsilon$) and $1/m_{\varepsilon_0}$ are $L^p$-multipliers ($1<p<\infty$), and consequently

$$\Big\|\sum_{j=0}^\infty 2^{j\alpha}\Phi_j * f\Big\|_{L^p} \approx \Big\|\sum_{\nu=0}^\infty (\nu+1)^\alpha \mathsf{P}_\nu * f\Big\|_{L^p} \tag{7.4}$$

and

$$\Big\|\sum_{j=0}^\infty \varepsilon_j 2^{j\alpha}\Phi_j * f\Big\|_{L^p} \le c\Big\|\sum_{\nu=0}^\infty (\nu+1)^\alpha \mathsf{P}_\nu * f\Big\|_{L^p} \tag{7.5}$$

for any $\varepsilon_j = \pm 1$. Then a routine argument using Khintchin's inequality involving Rademacher functions yields

$$\Big\|\Big(\sum_{j=0}^\infty (2^{\alpha j}|\Phi_j * f|)^2\Big)^{1/2}\Big\|_{L^p} \le c\Big\|\sum_{\nu=0}^\infty (\nu+1)^\alpha \mathsf{P}_\nu * f\Big\|_{L^p}, \tag{7.6}$$

i.e. $\|f\|_{F_p^{\alpha 2}} \le c\|f\|_{H_\alpha^p}$.

To prove the estimate in the other direction, we denote

$$g_\mu := \sum_{j\in\mathbb{Z}_\mu^+} 2^{j\alpha}\Phi_j * f \text{ with } \mathbb{Z}_\mu^+ := \{4\ell+\mu : \ell = 0,1,\dots\}\setminus\{0\}, \;\; \mu = 0,1,2,3.$$

Let $\widehat{b}(t) := \widehat{a}(t/2) + \widehat{a}(t) + \widehat{a}(2t)$. Evidently, $\widehat{b} \in C^\infty$, $\operatorname{supp} \widehat{b} \subset [1/4, 4]$, and $\widehat{b}(t) = 1$ if $t \in [1/2, 2]$. Write $m_{\varepsilon,\mu}(t) := \sum_{j\in\mathbb{Z}_\mu^+} \varepsilon_j \widehat{b}\left(\frac{t}{2^{j-1}}\right)$. If we denote

$$\sum_{j\in\mathbb{Z}_\mu^+} \varepsilon_j 2^{j\alpha} \Phi_j * f =: \sum_\nu h_\nu \mathsf{P}_\nu * f,$$

then

$$T_\varepsilon g_\mu := \sum_{j\in\mathbb{Z}_\mu^+} \varepsilon_j 2^{j\alpha} \Phi_j * f = \sum_\nu m_{\varepsilon,\mu}(\nu) h_\nu \mathsf{P}_\nu * f.$$

It is easy to see that $\|t^r m_{\varepsilon,\mu}^{(r)}(t)\|_{L^\infty} \le c(r) < \infty$ for $r = 0, 1, \dots$ and hence $m_{\varepsilon,\mu}$ is an $L^p$-multiplier (see [16]). Therefore, $\|T_\varepsilon g_\mu\|_{L^p} \le c\|g_\mu\|_{L^p}$. On the other hand, from the definition of $g_\mu$ and $T_\varepsilon g_\mu$, we have $T_\varepsilon^2 g_\mu = g_\mu$. Therefore,

$$\|g_\mu\|_{L^p} \le c\|T_\varepsilon g_\mu\|_{L^p} = c\Big\| \sum_{j\in\mathbb{Z}_\mu^+} \varepsilon_j 2^{j\alpha} \Phi_j * f \Big\|_{L^p}$$

for any $\varepsilon = \{\varepsilon_j\}$, $\varepsilon_j = \pm 1$. Now again the well-known argument using Khintchin's inequality gives

$$\|g_\mu\|_{L^p} \le c\Big\| \Big( \sum_{j\in\mathbb{Z}_\mu^+} (2^{j\alpha}|\Phi_j * f|)^2 \Big)^{1/2} \Big\|_{L^p} \le c\|f\|_{F_p^{\alpha 2}}, \ \mu = 0, 1, 2, 3.$$

This along with (7.4) implies $\|f\|_{H_\alpha^p} \le \|\mathsf{P}_0 * f\|_{L^p} + \sum_{\mu=0}^3 \|g_\mu\|_{L^p} \le c\|f\|_{F_p^{\alpha 2}}$. $\square$

**Proof of Lemma 4.6.** For the proof of this lemma we need the following technical lemma.

**Lemma 7.1.** *Let $g \in \Pi_{2^j}(\mathbb{S}^n)$, $j \ge 0$ and let $k > 0$. Suppose $\xi_1, \xi_2 \in \mathbb{S}^n$ and $d(\xi_\nu, \eta) \le c^* 2^{-j}$ ($\nu = 1, 2$) for some $\eta \in \mathcal{X}_j$. Then*

$$|g(\xi_1) - g(\xi_2)| \le c2^j d(\xi_1, \xi_2) \sum_{\omega\in\mathcal{X}_j} \frac{|g(\omega)|}{(1 + 2^j d(\omega, \eta))^k}, \tag{7.7}$$

*where $c$ is independent of $g$, $j$, $\xi_1$, $\xi_2$, and $\eta$.*

**Proof.** Assuming that $\widehat{d} \ge 0$ is an admissible function of type (a) (see Definition 2.1), we define

$$\mathcal{K}_j := \sum_{\nu=0}^\infty \widehat{d}\Big(\frac{\nu}{2^j}\Big) \mathsf{P}_\nu, \quad j \ge 1. \tag{7.8}$$

Thus $\mathcal{K}_j$ is actually $\mathcal{K}_N$ from (2.6) and Lemma 2.4 with $N = 2^j$ and hence $\mathcal{K}_j$ has the properties of $\mathcal{K}_{2^j}$ from Lemmas 2.4 and 2.6. In particular, $\mathcal{K}_j * g = g$ since $g \in \Pi_{2^j}(\mathbb{S}^n)$, and $\mathcal{K}_j(\omega \cdot \bullet) g \in \Pi_{2^{j+2}}(\mathbb{S}^n)$. Therefore, using Corollary 2.9, we obtain

$$g(\xi) = \int_{\mathbb{S}^n} \mathcal{K}_j(\omega \cdot \xi) g(\xi)\, d\mu(\omega) = \sum_{\omega\in\mathcal{X}_j} c_\omega \mathcal{K}_j(\omega \cdot \xi) g(\omega), \quad \xi \in \mathbb{S}^n. \tag{7.9}$$

By Lemma 2.6, we have

$$|\mathcal{K}_j(\omega \cdot \xi_1) - \mathcal{K}_j(\omega \cdot \xi_2)| \le \frac{cd(\xi_1, \xi_2) 2^{j(n+1)}}{(1 + 2^j d(\omega, \eta))^k}.$$

Combining this with (7.9), we obtain

$$\begin{aligned}|g(\xi_1)-g(\xi_2)| &\le \sum_{\omega\in\mathcal{X}_j} c_\omega |\mathcal{K}_j(\omega\cdot\xi_1)-\mathcal{K}_j(\omega\cdot\xi_2)||g(\omega)| \\ &\le c2^j d(\xi_1,\xi_2)\sum_{\omega\in\mathcal{X}_j}\frac{|g(\omega)|}{(1+2^j d(\omega,\eta))^k},\end{aligned}$$

which completes the proof. □

We are now prepared to prove Lemma 4.6. Evidently, $a_\eta \le b_\eta + d_\eta$, where

$$d_\eta := \sup\{|g(\xi_1)-g(\xi_2)| : \xi_1,\xi_2 \in G_\eta^\diamond, d(\xi_1,\xi_2)\le c_1 2^{-j-r}\}$$

with $G_\eta^\diamond := \{\xi \in \mathbb{S}^n : d(\xi,\eta) \le c_1 2^{-j} + 2c_1 2^{-j-r}\}$. Here $c_1$ is the constant from (2.16). Note that $G_\omega \subset G_\eta^\diamond$ if $\omega \in \mathcal{X}_{j+r}$ and $G_\omega \cap G_\eta \ne \emptyset$.

Fix $\xi_1,\xi_2 \in G_\eta^\diamond$ with $d(\xi_1,\xi_2)\le c_1 2^{-j-r}$. Then by Lemma 7.1,

$$|g(\xi_1)-g(\xi_2)| \le c2^{-r}\sum_{\omega\in\mathcal{X}_j}\frac{|g(\omega)|}{(1+2^j d(\omega,\eta))^k}$$

and hence

$$d_\eta \le c2^{-r}\sum_{\omega\in\mathcal{X}_j}\frac{|g(\omega)|}{(1+2^j d(\omega,\eta))^k}.$$

From this and the definition of $d_\eta^*$ (see (4.8)), we infer

$$\begin{aligned} d_\eta^* &\le c2^{-r}\sum_{\sigma\in\mathcal{X}_j}\frac{1}{(1+2^j d(\sigma,\eta))^k}\sum_{\omega\in\mathcal{X}_j}\frac{|g(\omega)|}{(1+2^j d(\omega,\sigma))^k} \\ &\le c2^{-r}\sum_{\omega\in\mathcal{X}_j}|g(\omega)|\sum_{\sigma\in\mathcal{X}_j}\frac{1}{(1+2^j d(\sigma,\eta))^k(1+2^j d(\omega,\sigma))^k} \\ &\le c2^{-r}\sum_{\omega\in\mathcal{X}_j}\frac{|g(\omega)|}{(1+2^j d(\omega,\eta))^k} = c^\diamond 2^{-r} a_\eta^*,\end{aligned}$$

where we used Lemma 4.8. Therefore, $a_\eta^* \le b_\eta^* + c^\diamond 2^{-r} a_\eta^*$ with $c^\diamond$ independent of $r$. Selecting $r$ sufficiently large, we obtain $a_\eta^* \le c b_\eta^*$. The estimate in the other direction is trivial. □

**Proof of Lemma 4.7.** Let $\eta \in \mathcal{X}_j$. Define $\mathcal{X}_{\eta,0} := \{\omega \in \mathcal{X}_j : 2^j d(\omega,\eta) < c_1\}$ and $\mathcal{X}_{\eta,m} := \{\omega \in \mathcal{X}_j : c_1 2^{m-1} \le 2^j d(\omega,\eta) < c_1 2^m\}$, $m = 1,2,\dots$, where $c_1$ is from (2.16). Evidently, $\#\mathcal{X}_{\eta,m} \approx 2^{mn}$. Write

$$\mathbb{G}_{\eta,m} := \{\xi \in \mathbb{S}^n : d(\xi,\eta) \le c_1(2^m+1)2^{-j}\}.$$

Clearly, $\bigcup_{\omega\in\mathcal{X}_{\eta,m}} G_\omega \subset \mathbb{G}_{\eta,m}$ and $|\mathbb{G}_{\eta,m}| \approx 2^{(m-j)n}$ if $m \le cj$. Set $\gamma := \min\{1, 1/s\}$. Then

$$\begin{aligned}
\sum_{\omega\in\mathcal{X}_{\eta,m}} \frac{|b_\omega|}{(1+2^j d(\omega,\eta))^k} &\le c2^{-mk} \sum_{\omega\in\mathcal{X}_{\eta,m}} |b_\omega| \le c2^{-mk+mn(1-\gamma)} \Big(\sum_{\omega\in\mathcal{X}_{\eta,m}} |b_\omega|^s\Big)^{1/s} \\
&\le c2^{-mk+mn(1-\gamma)+mn/s} \left(\frac{1}{|\mathbb{G}_{\eta,m}|}\int_{\mathbb{G}_{\eta,m}} \Big(\sum_{\omega\in\mathcal{X}_{\eta,m}} |b_\omega|\mathbb{1}_{G_\omega}(\xi)\Big)^s\right)^{1/s} \\
&\le c2^{-m(k-n(1+1/s-\gamma))}\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_{\eta,m}} |b_\omega|\mathbb{1}_{G_\omega}\Big)(\xi) \\
&\le c2^{-m(k-n\max\{1,1/s\})}\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_j} |b_\omega|\mathbb{1}_{G_\omega}\Big)(\xi), \quad \xi\in G_\eta,
\end{aligned}$$

where for the first estimate we used Hölder's inequality if $s>1$ and the $s$-triangle inequality if $s\le 1$. Summing over $m=0,1,\dots$ we obtain (4.10). □

**Proof of Lemma 4.8.** Let $\xi,\eta\in\mathcal{X}_j$ and $\xi\ne\eta$. We denote

$$\mathcal{S}_\xi := \{\sigma\in\mathbb{S}^n : d(\sigma,\xi)\ge d(\xi,\eta)/2\} \quad\text{and}\quad \mathcal{S}_\eta := \{\sigma\in\mathbb{S}^n : d(\sigma,\eta)\ge d(\xi,\eta)/2\}.$$

Evidently $\mathbb{S}^n = \mathcal{S}_\xi\cup\mathcal{S}_\eta$. We have

$$\begin{aligned}
\int_{\mathcal{S}_\xi} \frac{d\mu(\sigma)}{(1+2^j d(\xi,\sigma))^k(1+2^j d(\eta,\sigma))^k} &\le \frac{c}{(1+2^j d(\xi,\eta))^k}\int_{\mathbb{S}^n}\frac{d\mu(\sigma)}{(1+2^j d(\eta,\sigma))^k} \\
&\le \frac{c2^{-jn}}{(1+2^j d(\xi,\eta))^k}.
\end{aligned}$$

We similarly estimate the integral over $\mathcal{S}_\eta$ and then (4.11) follows.

To prove (4.12), we observe that $G_\eta := B_\eta(c_1 2^{-j})$ and $|G_\eta|\approx 2^{-jn}$, $\eta\in\mathcal{X}_j$ (see (2.16)). Then

$$(1+2^j d(\xi,\eta))^{-k} \le c(k,n)\inf_{\sigma\in G_\eta}(1+2^j d(\xi,\sigma))^{-k} \quad\text{for } \xi\in\mathbb{S}^n,\ \eta\in\mathcal{X}_j$$

and hence

$$\sum_{\sigma\in\mathcal{X}_j}\frac{2^{-jn}}{(1+2^j d(\xi,\sigma))^k(1+2^j d(\eta,\sigma))^k} \le c\int_{\mathbb{S}^n}\frac{d\mu(\sigma)}{(1+2^j d(\xi,\sigma))^k(1+2^j d(\eta,\sigma))^k}.$$

Now, (4.12) follows by (4.11). □

**Proof of Lemma 4.9.** Note first that the right-hand-side equivalence in (4.13) is immediate from (2.20). The estimate

$$\mathcal{M}_s\psi_\eta(\xi) \le \frac{c2^{jn/2}}{(1+2^j d(\xi,\eta))^{n/s}}, \quad \xi\in\mathbb{S}^n.$$

follows easily by (3.11).

We next show that

$$\mathcal{M}_s\psi_\eta(\xi) \ge \frac{c2^{jn/2}}{(1+2^j d(\xi,\eta))^{n/s}}, \quad \xi\in\mathbb{S}^n. \tag{7.10}$$

By (3.13), $\|\psi_\eta\|_{L^\infty(B_\eta(c_1^\diamond 2^{-j}))} \ge c_2^\diamond 2^{jn/2}$. Let $\omega\in B_\eta(c_1^\diamond 2^{-j})$ be such that

$$|\psi(\omega)| = \|\psi_\eta\|_{L^\infty(B_\eta(c_1^\diamond 2^{-j}))} \ge c_2^\diamond 2^{jn/2}. \tag{7.11}$$

We now claim that there exists a constant $\tilde{c} > 0$ such that

$$|\psi(\xi)| \geq (c_2^\diamond/2)2^{jn/2}, \quad \text{if } d(\omega,\xi) \leq \tilde{c}2^{-j}. \tag{7.12}$$

Indeed, evidently

$$\sum_{\omega\in\mathcal{X}_j} \frac{1}{(1+2^j d(\xi,\eta))^k} \leq c\int_{\mathbb{S}^n} \frac{2^{jn}}{(1+2^j d(\xi,\eta))^k}\, d\mu(\xi) \leq c < \infty.$$

We apply Lemma 7.1 to $\psi_\eta \in \Pi_{2^j}(\mathbb{S}^n)$ using the above and that $\|\psi_\eta\|_{L^\infty} \approx 2^{in/2}$ (see (3.12)) to obtain

$$|\psi_\eta(\xi) - \psi_\eta(\eta)| \leq c2^{j(n/2+1)} d(\xi,\eta), \quad \xi \in \mathbb{S}^n.$$

This and (7.11) readily imply (7.12) for sufficiently small $\tilde{c} > 0$.

Using (7.12) and (2.20) we obtain

$$\mathcal{M}_s\psi_\eta(\xi) \geq (c_2^\diamond/2)2^{jn/2}\mathcal{M}_s\mathbb{1}_{B_\omega(\tilde{c}2^{-j})}(\xi) \geq \frac{c2^{jn/2}}{(1+2^j d(\xi,\omega))^{n/s}} \geq \frac{c2^{jn/2}}{(1+2^j d(\xi,\eta))^{n/s}},$$

where we also used that $d(\omega,\eta) \leq c_2^\diamond 2^{-j}$. Thus (7.10) holds and this completes the proof of the lemma. □

**Proof of Lemma 5.7.** Fix $0 < s < \min\{p,1\}$ and $k > n/s$. We introduce the notation:

$$a_\eta := \sup_{\xi\in G_\eta} |g(\xi)|, \; m_\eta := \inf_{\xi\in G_\eta} |g(\xi)|, \; b_\eta := \max\{m_\omega : \omega \in \mathcal{X}_{j+r}, G_\omega \cap G_\eta \neq \emptyset\},$$

where $r \geq 1$ is from Lemma 4.6. We use Lemmas 4.6-4.7 and the maximal inequality (2.19) (for a single function) to obtain

$$\begin{aligned}\Big(\sum_{\eta\in\mathcal{X}_j}\sup_{\xi\in G_\eta}|g(\xi)|^p\Big)^{1/p} &\leq c2^{jn/p}\Big\|\sum_{\eta\in\mathcal{X}_j} a_\eta\mathbb{1}_{G_\eta}\Big\|_{L^p} \leq c2^{jn/p}\Big\|\sum_{\eta\in\mathcal{X}_j} b_\eta^*\mathbb{1}_{G_\eta}\Big\|_{L^p}\\ &\leq c2^{jn/p}\Big\|\mathcal{M}_s\Big(\sum_{\eta\in\mathcal{X}_j} b_\eta\mathbb{1}_{G_\eta}\Big)\Big\|_{L^p} \leq c2^{jn/p}\Big\|\sum_{\eta\in\mathcal{X}_j} b_\eta\mathbb{1}_{G_\eta}\Big\|_{L^p}.\end{aligned}$$

Now, exactly as in the proof of Theorem 4.5, we have

$$b_\eta\mathbb{1}_{G_\eta}(\xi) \leq c\sum_{\omega\in\mathcal{X}_{j+r}(\eta)} m_\omega^*\mathbb{1}_{G_\omega}(\xi),$$

where $\mathcal{X}_{j+r}(\eta) := \{\omega \in \mathcal{X}_{j+r} : G_\omega \cap G_\eta \neq \emptyset\}$. Note that $\#\mathcal{X}_{j+r}(\eta) \leq c(r,n)$. Using this, Lemma 4.7, and (2.19), we infer

$$\begin{aligned}\Big\|\sum_{\eta\in\mathcal{X}_j} b_\eta\mathbb{1}_{G_\eta}\Big\|_{L^p} &\leq c\Big\|\sum_{\omega\in\mathcal{X}_{j+r}} m_\omega^*\mathbb{1}_{G_\omega}\Big\|_{L^p} \leq c\Big\|\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_{j+r}} m_\omega\mathbb{1}_{G_\omega}\Big)\Big\|_{L^p}\\ &\leq c\Big\|\sum_{\omega\in\mathcal{X}_{j+r}} m_\omega\mathbb{1}_{G_\omega}\Big\|_{L^p} \leq c\|g\|_{L^p}\end{aligned}$$

and the lemma follows. □

**Proof of inequality (6.7).** We shall use the obvious inequality

$$a^\alpha b^{s-\alpha} \leq (a+b)^s, \quad \text{if } 0 < \alpha \leq s \quad \text{and} \quad a,b > 0, \tag{7.13}$$

which is immediate from $(a/b)^\alpha \le (a/b+1)^\alpha \le (a/b+1)^s$. Now, set $\alpha := 1/\tau - 1/p$, $s := 1/\tau > \alpha$, $a := m x_m^\tau$, and $b := \sum_{j=m+1}^\infty x_j^\tau$. Applying inequality (7.13), we find

$$\Big(\sum_{j=m+1}^\infty x_j^p\Big)^{1/p} \le \Big(x_m^{p-\tau}\sum_{j=m+1}^\infty x_j^\tau\Big)^{1/p} = x_m^{1-\tau/p}\Big(\sum_{j=m+1}^\infty x_j^\tau\Big)^{1/p}$$

$$= m^{-\alpha}a^\alpha b^{1/\tau-\alpha} \le m^{-\alpha}(a+b)^{1/\tau} \le m^{-\alpha}\Big(\sum_{j=1}^\infty x_j^\tau\Big)^{1/\tau}.$$

□

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