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The series $\sum\sum \frac{e^{2\pi i m n x}}{mn}$, and a problem of Chowla

K.I. Oskolkov

Department of Mathematics
University of South Carolina

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Abstract

The double trigonometric series with the hyperbolic phase

$$U(x) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi imnx}}{\pi mn}$$

is studied. Complete descriptions of the $\mho$-convergence (summability) sets of the sin-series $\Im U(x)$, and the cos-series $\Re U(x)$ are established. The $\mho$-sum of a double series is defined as the common value of the limits of the partial sums over expanding families of coordinate-wise convex domains in $\mathbb{N}^2$. The latter incorporate convex domains in the usual sense, such as rectangles, but also, say, the *hyperbolic crosses* $\diamondsuit_N := \{(m,n):\ 1 \le mn \le N\}$. In particular, let $q_j(x)$ denote the denominator of the j-th *convergent* of x, $\mathrm{frac}(x)$ – the fractional part of x, and $d(n) := \sum_{d|n} 1$ – the divisor function. Then for every real x, the series $\Im U(x)$ is $\mho$-convergent, or divergent, simultaneously with each of the following three series:

$$\Xi(x) := \sum_{j=1}^{\infty} \frac{(-1)^j \ln q_{j+1}(x)}{q_j(x)},\ S(x) := \sum_{n=1}^{\infty} \frac{1/2 - \mathrm{frac}(nx)}{n},\ T(x) := \sum_{n=1}^{\infty} d(n)\frac{\sin 2\pi nx}{\pi n}.$$

Besides, the equality $S = T$ is true for each x where Ξ converges. These claims provide the solution of a problem raised by S. D. Chowla in 1931 (see [3]).
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1 Introduction

We will say that a set Ω in the first quadrant $\mathbb{R}^2_+$ of the real plane $\mathbb{R}^2$ is a *coordinate-wise convex domain*, iff there exist a non-negative and monotonically decreasing function $y = f(x),\ x > 0$ such that

$$\Omega = \bigcup_{x>0} \{(x,y):\ 0 < y \le f(x)\}.$$

Let us call such a set $\mho$-*domain*, and use the notation $\mho$ also for the whole class of all $\mho$-domains. For $\Omega \in \mho$ denote $M(\Omega)$ the side-length of the biggest square $[1, M] \times [1, M]$ contained in Ω, i. e. $M(\Omega) := \sup\{M : \quad [1, M] \times [1, M] \subset \Omega\}$. We say that a (countable) family of $\mho$-domains $\{\Omega_r\}_{r=1}^{\infty}$ is *expanding* iff $M(\Omega_r) \to \infty,\ r \to \infty$. If, in addition, for each sufficiently large $q \in \mathbb{R}$, there exist such $r = r(q)$ that $\frac{q}{2} \leq M(\Omega_r) < q$, we say that this family is *full*. Further, we say that a double-indexed numerical sequence $u = \{u_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ is $\mho$-summable, iff the limit

$$(\mho) \sum_{(m,n)\in\mathbb{N}^2} u_{m,n} := \lim_{r\to\infty} U_{\Omega_r}; \quad U_\Omega := \sum_{(m,n)\in\Omega} u_{m,n}$$

exists for each expanding family $\{\Omega_r\}_{r=1}^{\infty} \subset \mho$. It is easy to see that if u is $\mho$-summable, then the value of the $\mho$-sum is unique, i. e., the limit is the same for all expanding families of $\mho$ domains.

Characteristic functions $\chi = \{\chi_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of $\mho$ domains obviously possess a uniformly bounded coordinate-wise variation (notation: CBV), i. e. $\chi_{m,n} = \chi(m, n)$, where $\chi : \mathbb{R}_+^2 \mapsto \mathbb{C}$ and

$$\|\chi\|_{\mathrm{CBV}} := \sup_{\mathbb{R}_+^2} |\chi| + \sup_{x>0} \left(\mathrm{var}_{R_+}\chi(\cdot, x) + \mathrm{var}_{R_+}\chi(x, \cdot)\right) < \infty.$$

CBV-sequences constitute a subclass of a wider class of *coordinate-wise slow multipliers* (notation: CS). A uni-variate function $\chi : \mathbb{R}_+ \mapsto \mathbb{C}$ is by definition *slow*, if it satisfies the Littlewood – Paley condition (cf. [29], Ch. 15):

$$\|\chi(\cdot)\|_{\mathrm{CS}} := \sup_{x>0} \left(|\chi(x)| + \mathrm{var}_{[x,2x]}\, \chi\right) < \infty,$$

and uni-variate CS-sequences are simply restrictions of slow functions onto the set of natural numbers. A bi-variate sequence $\chi : \mathbb{N}_+^2 \mapsto \mathbb{C}$ is called coordinate-wise slow, if $\chi_{m,n} = \chi(m, n)$ where the function $\chi : \mathbb{R}_+^2 \mapsto \mathbb{C}$ satisfies

$$\|\chi(\cdot)\|_{\mathrm{CS}} := \sup_{x>0} (\|\chi(\cdot, x)\|_{\mathrm{CS}} + \|\chi(x, \cdot)\|_{\mathrm{CS}}) < \infty.$$

It is easy to see that if $\chi^{(1)}, \chi^{(2)} \in \mathrm{CS}$ then the product sequence $\chi := \chi^{(1)} \cdot \chi^{(2)}$ is also coordinate-wise slow. Further, "zeta-like" multipliers $\chi_{m,n} = m^{it_1} n^{it_2}$, or $\chi_{m,n} = (m^2 + n^2)^{it}$, where $t,\ t_1,\ t_2$ are fixed real numbers, are examples of coordinate-wise slow sequences, that do not belong to CBV.

Our goal is to study the $\mho$-convergence of the double trigonometric series

$$U(x) := \sum_{(m,n)\in\mathbb{N}^2} \frac{e^{2\pi imnx}}{\pi mn}, \quad U(\chi, x) := \sum_{(m,n)\in\mathbb{N}^2} \chi_{m,n} \frac{e^{2\pi imnx}}{\pi mn}, \quad \chi \in \mathrm{CS}.$$

Given a (bounded) $\mho$-domain, a sequence $\chi \in \mathrm{CS}$, and $x \in \mathbb{R}$, denote:

$$U_\Omega(\chi, x) := \sum_{(m,n)\in\Omega} \chi_{m,n} \frac{e^{2\pi mnx}}{\pi mn}\,, \quad U_\Omega(\{1\}, x) := U_\Omega(x)\,;$$

$\mathrm{frac}(x)$ – the fractional part of $x \in \mathbb{R}$, and $\mathrm{frac}(0) := 0$; $\Im z$, $\Re z$ – respectively, the imaginary, and the real part of a complex number z; $\mathrm{sign}\, z := \frac{z}{|z|}$, $z \neq 0$; $|I|$ – the length of an interval $I \subset \mathbb{R}$; the constants in the symbols $O, \ll$ are absolute;
$\mathbb{Q}$ – the set of all rational numbers;
$d(n) := \sum_{d|n} 1$, $n \in \mathbb{N}$ – the divisor function;

$$x = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}} = [k_1, k_2, \ldots], \quad \frac{a_j}{q_j} := [k_1, k_2, \ldots, k_j], \quad j = 1, 2, \ldots \tag{1}$$

– the simple continued fraction for $x \in (0,1)$ (cf. e. g. [14], Chapter 10, or [28], Chapter 1); the natural numbers $k_j = k_j(x)$ are known as *partial quotients* of x, and the rational numbers $\frac{a_j}{q_j} = \frac{a_j}{q_j}(x)$ – the *convergents* of x; $q_0(x) := 1$;

$$\Xi(x) := \sum_{j=0}^{\infty} \frac{\ln^2 q_{j+1}}{2\pi q_j} + i \sum_{j=0}^{\infty} \frac{(-1)^j \ln q_{j+1}}{2q_j}, \quad \Upsilon(x) := \sum_{j=0}^{\infty} \frac{\ln q_{j+1}}{q_j} \quad q_j = q_j(x);$$

$\mathbb{D}(\cdot)$ denotes the divergence ($\mho$-divergence) set of the series, or the sequence in ( );
F_j denotes the j-th Fibonacci number, i. e. $F_0 = F_1 := 1$, $F_{j+1} = F_j + F_{j-1}$, $j = 1, 2, \ldots$

Theorem 1 *A. The $\mho$-divergence sets of the imaginary and the real parts of the series U are described by the relations*

$$\mathbb{D}(\Im U) = \mathbb{D}(\Im \Xi), \quad \mathbb{D}(\Re U) = \mathbb{Q} \cup \mathbb{D}(\Re \Xi),$$

and point-wise $U(x) = \Xi(x) + f(x)$, where $|\Im f| \ll 1$, $|\Re f| \ll \Upsilon$.

Further, for every full sequence of $\mho$ domains $\{\Omega_r\}_{r=1}^{\infty}$

$$\mathbb{D}(\Im \Xi) \subset \mathbb{D}(\{\Im U_{\Omega_r}\}), \quad (\mathbb{D}(\Re \Xi) \cup \mathbb{Q}) \subset \mathbb{D}(\{\Re U_{\Omega_r}\})\,.$$

If $\chi \in \mathrm{CS}$, $\|\chi\|_{\mathrm{CS}} \leq 1,$, $\Im \chi \equiv 0$, then $\mathbb{D}(\Im U(\chi)) \subset \mathbb{D}(\Upsilon)$, $\mathbb{D}(U(\chi)) \subset \mathbb{Q} \cup \mathbb{D}(\Re \Xi)$, and point-wise $|\Im U(\chi, x)| \ll \Upsilon(x)$, $|\Re U(\chi, x)| \ll \Re \Xi(x)$.

The divergence sets $\mathbb{D}$ in this theorem are rather "thin". To quantify this property, e. g. for the series Υ, let us apply two following characteristics: 1) the estimates of the remainder terms of Υ

$$\Upsilon_j(x) := \sum_{k=j}^{\infty} \frac{\ln q_{k+1}}{q_k}\,, \quad j = 0, 1, \ldots;$$

2) the estimates of the "logarithmic dimension" of $\mathbb{D}(\Upsilon)$.

Theorem 2 *A. The function Υ is exponentially integrable:*

$$\int_0^1 \exp(\lambda\Upsilon(x))\,dx < \infty, \quad 0 < \lambda < 1.$$

If a sequence of positive numbers λ_j satisfies the condition $\lambda_j = o(F_j/j)$, $j \to \infty$, then

$$\int_0^1 \left(\exp(\lambda_j \Upsilon_j(x)) - 1\right)\,dx \to 0.$$

B. If $\alpha > 0$ and ε – an arbitrarily small positive number, then there exist a family of intervals $\mathcal{I} = \mathcal{I}(\alpha, \varepsilon) = \{I\}$ covering $\mathbb{D}(\Upsilon)$, and such that

$$\sum_{I \in \mathcal{I}(\alpha,\varepsilon)} |\ln |I||^{-\alpha} \le \varepsilon^{\alpha}.$$

In particular, the Hausdorff dimension of $\mathbb{D}(\Upsilon)$ equals 0.

Remark. Unlike the function $\Im U$, its trigonometric conjugate $\Re U$ is not exponentially integrable; moreover,

$$\int_0^1 \exp\left(\sqrt{|2\pi \Re U(x)|}\right)\,dx = +\infty.$$

(The latter easily follows from theorems 1 and 2 and the consideration of the function $\ln^2 q_1(x)$). Therefore, U is not a function of bounded mean oscillation, see e. g. [24], Ch. 4.

Before turning to the proof of these statements, let us deduce some corollaries, and provide comments. For $(M, N) \in \mathbb{N}^2$, let us consider the rectangles, infinite stripes, and the hyperbolic crosses $\diamondsuit$:

$$\square_{M,N} := [1, M] \times [1, N],\ \square_{\infty,N} := [1, \infty) \times [1, N],\ \diamondsuit_N := \{(m, n) \in \mathbb{N}^2 : 1 \le mn \le N\}.$$

Obviously, the domains $\square$ are convex in the usual sense, while the crosses $\diamondsuit$ are not convex examples of $\mho$ domains.

Also, recall that (cf. e. g. [29], Ch. 1)

$$L(y) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n y}}{\pi n} = -\frac{\ln|2\sin \pi y|}{\pi} + i\left(\frac{1}{2} - \operatorname{frac} y\right),$$

and apply to U the following two different particular $\mho$-summation methods:

$$\lim_{N\to\infty} \sum_{(m,n)\in\square_{\infty,N}} \frac{e^{2\pi i m n x}}{\pi m n} = -\sum_{m=1}^{\infty} \frac{\ln|2\sin \pi m x|}{\pi n} + i\sum_{m=1}^{\infty} \frac{1/2 - \operatorname{frac}(mx)}{m} := \tilde{S}(x) + iS(x),$$

$$\lim_{N\to\infty} \sum_{(m,n)\in\diamondsuit_N} \frac{e^{2\pi i m n x}}{\pi m n} = \lim_{N\to\infty} \left(\sum_{m\in[1,N]} d(m)\frac{\cos 2\pi m x + i \sin 2\pi m x}{\pi m}\right) := \tilde{T}(x) + iT(x).$$

The next statement is a direct corollary from theorem 1.

Corollary 1 *A. The convergence sets of the series $S, T, \Im\Xi$ coincide, and $S(x) = T(x)$ everywhere on the common convergence set.*
B. The convergence sets of the series $\tilde{S}, \tilde{T}$ coincide. The common convergence set consists of those irrational x, for which the series $\Re\Xi(x)$ is convergent; for such x, $\tilde{S}(x) = \tilde{T}(x)$.

The author's interest to the convergence problem of the series U was motivated by the recent papers [9], [20] –[22], and [16]. In [9] M. Z. Garaev studied the sequence $\{\Im U_{\square_{N,N}}\}$, i. e. the partial sums of the sin-series $\Im U$ over squares. He proved that $\mathbb{D}(\{\Im U_{\square_{N,N}}\}) \neq \emptyset$: the sequence diverges e. g. for

$$x = \sum_{j=1}^{\infty} \frac{1}{p_j}$$

where $p_1 := 2,\ p_{j+1} := p_j^{jp_j+1},\ j = 1, 2, \ldots$. From a personal communication with Garaev, the author learnt that the pointwise convergence problem of the series T was raised by S. D. Chowla. [2] shows that Chowla indeed addressed the problem in his publications [3] – [6] (see also [2], v. 1, pp. 230 – 249 and pp. 380 – 405; v. 2, pp. 489 – 491 and pp. 492 – 494). Particularly, in [3] it was proven that
a) if the sequence of the partial quotients $\{k_j(x)\}$ (see (1)) is bounded, then the series $S(x),\ T(x)$ converge, and the *Chowla's identity* (cf. [4]) $S(x) = T(x)$ is true;
b) there exist irrational numbers x for which $\tilde{T}(x)$ is divergent;
c) the divergence set of S is non-empty.
These results received further developments in [4] – [6]. However, the question whether or not there exist points x where the series $T(x)$ diverges, seemingly remained open. Corollary 1 provides the solution: $\mathbb{D}(T) = \mathbb{D}(S) = \mathbb{D}(\Im\Xi) \neq \emptyset$. Corollary 1 also contains a complete description of the sets where Chowla's identities $S(x) = T(x),\ \tilde{S}(x) = \tilde{T}(x)$ are valid.

Obviously, the sin-series $\Im U$ can be written in the form

$$\Im U(x) = \sum_{(n,m)\in\mathbb{Z}^2,\ nm\neq 0} \frac{e^{2\pi i m n x}}{4\pi i m n}.$$

Therefore, $\Im U$ is the simplest version of the multiple discrete oscillatory Hilbert transforms, with the multi-variate polynomial phase. (Here and in the sequel, we consider only algebraic polynomials P with the real coefficients; $\deg P$ denotes the degree of a polynomial P; the factors $A(\cdot)$ are finite numbers that depend only on the indicated parameters.)

The one-dimensional global boundedness result

$$\left| \sum_{n=1}^{r} \frac{e^{iP(n)} - e^{iP(-n)}}{n} \right| \leq A(\deg P),$$

including the convergence as $r \to \infty$, was proved in [1], and independently somewhat later – by Stein and Wainger, see [23]. In connection with this result, let us note Chowla's paper

[7] (see also [2], v. 1, pp. 426 –428), that discussed the *unboundedness* problem for the sums $\sum_1^r e^{i\alpha n^2}\frac{\sin n\beta}{n}$. A footnote on the first page of [7] mentions that the formulation of the latter problem is due to H. Davenport and H. Heilbronn. Secondly, it is remarked that the problem has been solved in the negative by Dr. Spaček of Prague. The latter amounts to the statement that the discrete Hilbert transforms with the polynomial phase of degree 2 are indeed uniformly bounded. However, the author of the present paper did not succeed in locating the associated publications.

One-dimensional trigonometric series with the polynomial phase

$$\sum_{n\in\mathbb{Z}^1} \hat{f}_n e^{2\pi i(n^d x_d+\cdots+nx_1)}$$

($x_d, \ldots, x_1$ are considered as real variables) have found applications in the study of the solutions of the Cauchy initial value problem for Schrödinger type equations, with the periodic initial data functions, cf. e. g. [17], [19].

Stein and Wainger [20] – [22] elaborated the operator properties of the multiple trigonometric series with the multi-variate polynomial phase. In [20] a fundamental theorem is established, stating the $l^2(\mathbb{Z}^k) \mapsto l^2(\mathbb{Z}^k)$ boundedness of the discrete oscillatory transforms with the multivariate polynomial phase

$$T(f)(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^k,\ \mathbf{m}\neq\mathbf{n}} e^{2\pi P(\mathbf{n},\mathbf{m})} K(\mathbf{n}-\mathbf{m}) f(\mathbf{m}), \quad \mathbf{n}\in\mathbb{Z}^k$$

where $P(\mathbf{n},\mathbf{m})$ is an algebraic polynomial on $\mathbb{Z}^k\times\mathbb{Z}^k$, K - a Calderón – Zygmund kernel. The norm of the operator $T:\ l^2(\mathbb{Z}^k)\mapsto l^2(\mathbb{Z}^k)$ does not exceed a number $A = A(\deg P, k)$. The proof of this result in [20] required elaboration of the estimates of the multivariate exponential sums of H. Weyl with the slow multipliers over convex domains $\Omega\subset\mathbb{R}^k$

$$w_\Omega(\chi, P) := \sum_{\mathbf{n}\in\Omega} \chi_{\mathbf{n}} e^{2\pi i P(\mathbf{n})}\,.$$

As a particular case, the following results are mentioned in [20] (see (9.4) on p. 1334): if K is a Calderón – Zygmund kernel, then

$$\left|\sum_{|\mathbf{n}|\leq r} K(\mathbf{n}) e^{2\pi i P(\mathbf{n})}\right| \leq A(\deg P),$$

and the limit of the sums exists, as $r\to\infty$, if at least one of the coefficients of P is irrational. It is also noted that if e. g. $K(\mathbf{x}) = |\mathbf{x}|^{-k+i\gamma}$, γ real, $\gamma\neq 0$, then the limit does not exist for infinitely many values of the coefficients of P.

Let us mention that *slowness* of a multiplier is understood in [20] (see p. 1305) in the following sense: $\chi(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^k$ is slow if it is a C^1 function bounded on $\mathbb{R}^k$ satisfying

$$|\chi(\mathbf{x})| + |\mathbf{x}||\nabla\chi(\mathbf{x})| \ll 1.$$

Apparently, this definition is not too distant from our definition of a CS-function. However, we demand the property of slowness coordinate-wise, which is a somewhat lighter restriction, and we also consider the summation over $\mho$-domains, such as hyperbolic crosses, that are not convex in the usual sense, and also the $\mho$-domains "with diadic gaps". The latter feature allows to derive from theorem 1 corollaries concerning the divisor function, and simultaneously, the distribution of the fractional parts. The following statement is an example.

Corollary 2 *If* $\chi:\ \mathbb{R}_+ \mapsto \mathcal{C}$ *is a slow function,* $||\chi||_{\mathrm{CS}} \le 1$*, then*

$$\text{(i)}\ \sum_{k=0}^{\infty} 2^{-k}\left|\sum_{m\in[2^k,2^{k+1})} \chi(m)d(m)e^{2\pi imx}\right| \ll \Re\Xi(x),\ x\in\mathbb{R}\setminus\mathbb{Q}; \quad ,$$

$$\text{(ii)}\ \sum_{k=0}^{\infty} 2^{-k}\left|\sum_{m\in[2^k,2^{k+1})} \chi(m)\left(1/2-\mathrm{frac}(mx)\right)\right| \ll \Upsilon(x),\ x\in\mathbb{R}.$$

Indeed, if $\{\theta_k\}_{k=0}^{\infty}$ is a sequence of complex numbers satisfying $|\theta_k|\le 1$, then the function $\Theta:\ \mathbb{R}_+\mapsto\mathcal{C}$ defined by

$$\Theta(y) := 0,\ y\in(0,1), \quad \Theta(y) := \theta_k 2^{-k}y,\ y\in[2^k,2^{k+1}),\ k=0,1,\ldots$$

is slow, and $||\Theta||_{\mathrm{CS}} \ll 1$. To deduce corollary 2 from theorem 1, we take, for a fixed $x\in\mathbb{R}$,

$$\theta_k := \mathrm{sign}\sum_{m\in[2^k,2^{k+1})}\chi(m)d(m)e^{2\pi imx}, \quad \tilde{\theta}_k := \mathrm{sign}\sum_{m\in[2^k,2^{k+1})}\chi(m)\left(1/2-\mathrm{frac}(mx)\right),$$

and apply the following general property of the class CS: if $\chi:\ \mathbb{R}_+\mapsto\mathcal{C}$ is a slow uni-variate function, then $\Theta(mn)\chi(mn), \Theta(m)\chi(m),\ (m,n)\in\mathbb{R}^2_+$ are slow bi-variate functions with the CS-norm $\ll 1$.

The summation of trigonometric Fourier series over $\mho$ domains was considered by S.A. Telyakovskii in [25]-[26] where the following global boundedness result was established for the multiple sin-sums with the linear phases:

$$\sup_{\Omega\in\mho}\sup_{\mathbf{x}\in\mathbb{R}^k}\left|\sum_{\mathbf{n}\in\Omega}\frac{\sin 2\pi n_1x_1}{n_1}\cdots\frac{\sin 2\pi n_kx_k}{n_k}\right| < \infty.$$

This result, and its subsequent generalizations, cf. e. g. [27], have found numerous applications: estimations of Kolmogorov's diameters of the functional classes with bounded mixed derivative, hyperbolic cross approximations, the convergence theory of multiple Fourier series of bounded variation in the sense of Hardy, etc.

The paper [16] contains a unification of Telyakoskii's result and Arkhipov – Oskolkov's [1]. The multiple discrete Hilbert transforms with the additive polynomial phase P, and CS-multipliers are considered. It is proved that

$$\left| \sum_{\mathbf{n}\in\mathbb{N}^k} \chi_{\mathbf{n}} \frac{e^{iP_1(n_1)} - e^{iP_1(-n_1)}}{n_1} \cdots \frac{e^{iP_k(n_k)} - e^{iP_k(-n_k)}}{n_k} \right| \leq A(d)\|\chi\|_{\mathrm{CS}}, \quad d := \max_k \deg P_k,$$

and the multiple series is $\mho$-summable. In particular, for the real t_1, t_2 one has

$$\sup_{\Omega\in\mho} \left| \sum_{(m,n)\in\Omega} \frac{e^{iP(m)} - e^{iP(-m)}}{m^{1+it_1}} \frac{e^{iQ(n)} - e^{iQ(-n)}}{n^{1+it_2}} \right| \leq A(d, t_1, t_2), \quad d := \max(\deg P, \deg Q).$$

In view of this result, and also theorem 1, it seems interesting to study the double sums of the type

$$W_\Omega(x, z_1, z_2) := \sum_{(m,n)\in\Omega} \frac{e^{2\pi i mnx}}{\pi m^{z_1} n^{z_2}}$$

where $z_1 = \sigma_1 + it_1$, $z_2 = \sigma_2 + it_2$ are fixed complex numbers. For $z_1 = 1$, the imaginary part of the full sum $W(x, 1, z)$ coincides (see also Corollary 2) with the Dirichlet's series

$$\mathcal{H}(z, x) := \sum_{n=1}^{\infty} \frac{1/2 - \mathrm{frac}(nx)}{n^z},$$

which was introduced by E. Hecke. Hecke [13], G.H. Hardy and J.E. Littlewood in [11], [12], see also [10], pp. 197 – 252, studied $\mathcal{H}(z, x)$ as a function of the complex variable z *for fixed* x.

Recently, the author [18] studied a modification of the series U with "a somewhat bigger denominator" and CS-multipliers

$$V(\chi, x) := \sum_{(m,n)\in\mathbb{N}^2} \chi_{m,n} \frac{\sin 2\pi mnx}{m^2 + n^2}.$$

Unlike U, the series V converge for all real x, and the sums are bounded functions of x.

Let us also note that the double oscillatory sums with the hyperbolic phase, of the type

$$\sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}\, e^{2\pi i(mnt + mx_1 + nx_2)}$$

(t, x_1, x_2 – real variables) naturally appear in the study of the probabilistic density functions generated by the solutions of the Schrödinger equation with the periodic initial data, and also the boundary value problem posed for Helmholtz equation. Some results in this direction have been recently obtained in [19].

2 The proof

Proof of Theorem 1. Let Ω be a $\mho$ domain. As above, denote $M(\Omega) := M$ the side-length of the largest square $[1, M] \times [1, M]$ contained in Ω. For an expanding family $\{\Omega_r\} \subset \mho$ we have $M(\Omega_r) \to \infty$, so that without loss of generality we can assume that $M \geq 1$. Everywhere below we assume that the multiplier χ is a *real-valued* CS- sequence, and $\|\chi\|_{\mathrm{CS}} \leq 1$. For $x \in \mathbb{R}$, $\|x\|$ denotes the distance from x to the nearest integer. As above, we drop χ as the argument of the sums, if $\chi \equiv 1$.

Let us split the sum

$$U_\Omega(\chi, x) = \sum_{(m,n)\in\Omega} \chi_{m,n} \frac{e^{2\pi i m n x}}{\pi m n}$$

along the main diagonal $m = n$, and write, using our definitions of a $\mho$ domain and the number $M = M(\Omega)$:

$$U_\Omega(\chi, x) = \sum_{m\in[1,M]} \frac{\sigma_m(\chi, mx)}{m}\,, \quad \sigma_m(\chi, \Omega, y) := \sum_{n=m}^{N_m} \chi_{m,n} \frac{e^{2\pi i n y}}{\pi n} + \sum_{n=m+1}^{M_m} \chi_{n,m} \frac{e^{2\pi i n y}}{\pi n}\,,$$

where $N_m \geq M$, $M_m \geq M$, $\quad m \in [1, M]$. For $m \in \mathbb{N}$, let us denote

$$\sigma_{m,*}(\chi, y) := \sup_{N\geq m} \left| \sum_{n=m}^{N} (\chi_{m,n} + \chi_{n,m})' \frac{e^{2\pi i n y}}{\pi n} \right|\,, \quad \sigma_{m,\infty}(\chi, y) := \sum_{n=m}^{\infty} (\chi_{m,n} + \chi_{n,m})' \frac{e^{2\pi i n y}}{\pi n}\,,$$

where $'$ denotes that the term with $n = m$ has to be taken with the factor $1/2$. For all $m \in \mathbb{N}$ the series $\sigma_{m,\infty}(\chi, y)$ are convergent, unless y is an integer; $\Im\sigma_{m,\infty}(\chi, y)$ converge for all real y (see also lemma 1 below).

Let us temporarily assume that $x \in (0, 1)$ is an irrational number, denote $q_0 := 1$, and subdivide the summation domain $[1, M]$ into the intervals of the form $[q_j, q_{j+1})$, where $q_j = q_j(x)$ are the denominators of the continued fraction (1), $J = J(x, \Omega) := \max\{j : \ q_j \leq M\}$:

$$U_\Omega(\chi, x) = \sum_{j=0}^{J} A_j(\chi, \Omega, x)\,, \quad A_j(\chi, \Omega, x) := \sum_{m\in[q_j, q_{j+1})} \frac{\sigma_m(\chi, \Omega, mx)}{m}\,, \quad j = 0, \ldots, J-1,$$

$$A_J(\chi, \Omega, x) := \sum_{m\in[q_J, M]} \frac{\sigma_m(\chi, \Omega, mx)}{m}\,.$$

Let us introduce the following upper bounds for $A_j(\chi,\Omega,x)$:

$$A_{j,*}(\chi,x) := \sum_{m\in[q_j,q_{j+1})} \frac{1}{m} \sup_{\Omega\in\mho} |\sigma_m(\chi,\Omega,x)| = \sum_{m\in[q_j,q_{j+1})} \frac{\sigma_{m,*}(\chi,mx)}{m}\,.$$

Lemma 1 *For $m\in\mathbb{N},\ y\in\mathbb{R}$, the following estimates are true*

$$\text{(i) } \sigma_{m,*}(\chi,y) = O\left(\ln\left(1+\frac{1}{m\|y\|}\right)\right)\,,\quad \text{(ii) } \Im\sigma_{m,*}(\chi,y) = O\left(\min\left(1,\frac{1}{m\|y\|}\right)\right)\,;$$
$$\text{(iii) } \Im\sigma_m(\Omega,y) = \operatorname{sign} y + O\left(m|y| + \min\left(1,\frac{1}{M|y|}\right)\right)\,,$$
$$\text{(iv) } \Re\sigma_m(\Omega,y) = \frac{2}{\pi}\ln\frac{1}{m|y|} + O\left(1+(my)^2+\frac{1}{M|y|}\right)\,,\ |y|\le\frac{1}{2}\,,\ m\in[1,M). \tag{2}$$

Lemma 2 *For $x\in(0,1)\setminus\mathbb{Q}$*

$$\text{(i) } A_{j,*}(\chi,x) = O\left(\frac{\ln^2 q_{j+1}}{q_j}\right)\,,\quad \text{(ii) } \Im A_{j,*}(\chi,x) = O\left(\frac{\ln q_{j+1}}{q_j}\right)\,,\ j\in\mathbb{Z}_+;$$
$$\text{(iii) } A_j(\Omega,x) = \frac{\ln^2 q_{j+1}}{2\pi q_j} + i(-1)^j\frac{\ln q_{j+1}}{2q_j} + \varepsilon_j(\Omega,x),$$
$$\Im\varepsilon_j(\Omega,x) = O\left(\frac{\ln q_j}{q_j}\right)\,,\quad \Re\varepsilon_j(\Omega,x) = O\left(\frac{\ln q_{j+1}+\ln^2 q_j}{q_j}\right)\,,\ 0\le j\le J-1, \tag{3}$$

and if $M(\Omega)\ge\frac{q_{J+1}}{2}$, then the estimates (3,iii) remain true for $j=J$.

Proof of the lemmas. For $m,N\in\mathbb{N},\ y\in\mathbb{R}$ and a CS-sequence $\{\chi_n\}_{n\in\mathbb{N}}$ let

$$D_m(y) := \frac{1}{\pi m}\sum_{n=1}^{m} e^{2\pi i n y}\,,\quad F_{m,N}(\chi,y) := \sum_{n=m}^{N}\chi_n\frac{e^{2\pi i n y}}{\pi n}\,,$$
$$F_{m,*}(\chi,y) := \sup_{N\ge m}|F_{m,N}(\chi,y)|\,,\quad F_{m,\infty}(\chi,y) := \lim_{N\to\infty} F_{m,N}(\chi,y)\,.$$

We have, applying the Abel's transformation,

$$F_{m,N}(\chi,y) = \sum_{n=m}^{N}\chi_n\left(D_n(y) - \left(1-\frac{1}{n}\right)D_{n-1}(y)\right)$$
$$= \sum_{n=m}^{N}\chi_n\frac{D_{n-1}(y)}{n} - \chi_m D_{m-1}(y) + \sum_{n=m}^{N}(\chi_n-\chi_{n+1})D_n(y)\ + \chi_{N+1}D_N(y),$$

and therefore

$$F_{m,*}(\chi, y) \leq \sum_{n=m}^{\infty} \left(\frac{|\chi_n D_{n-1}(y)|}{n} + |(\chi_n - \chi_{n+1}) D_n(y)| \right) + 2 \sup_{n \geq m} |\chi_n D_{n-1}(y)|.$$

From here and the well-known estimate of the Dirichlet's kernel

$$D_n(y) = O\left(\min\left(1, \frac{1}{n\|y\|}\right)\right)$$

we conclude that

$$\sum_{n=m}^{\infty} \frac{|\chi_n D_{n-1}(y)|}{n} \ll \sup_{\nu\in\mathbb{N}} |\chi_\nu| \sum_{n=m}^{\infty} \frac{1}{n} \min\left(1, \frac{1}{n\|y\|}\right) = O\left(\ln\left(1 + \frac{1}{m\|y\|}\right)\right),$$

$$\sum_{n=m}^{\infty} |(\chi_n - \chi_{n+1}) D_n(y)| \leq \sum_{k=0}^{\infty} \left(\max_{n\in[2^k m, 2^{k+1} m)} |D_n(y)| \right) \mathrm{var}_{[2^k m, 2^{k+1} m)} \chi$$

$$\leq \sum_{k=0}^{\infty} \min\left(1, \frac{1}{2^k m\|y\|}\right) = O\left(\ln\left(1 + \frac{1}{m\|y\|}\right)\right),$$

$$\sup_{N\geq m} |\chi_{N+1} D_N(y)| = O\left(\min\left(1, \frac{1}{m\|y\|}\right)\right).$$

Summarizing these estimates we see that

$$F_{m,*}(\chi, y) = O\left(\ln\left(1 + \frac{1}{m\|y\|}\right)\right).$$

This relation implies the convergence of the series $F_{m,\infty}(\chi, y)$, $\sigma_{m,\infty}(\chi, y)$ for $y \in \mathbb{R} \setminus \mathbb{Z}$, and also the validity of the estimates (2,i).

The proof of (2,ii) follows a completely analogous path, with some modifications due to a better estimate of the imaginary part of the Dirichlet's kernel D_n:

$$\Im D_n(y) = O\left(\min\left(n\|y\|, \frac{1}{n\|y\|}\right)\right),$$

$$\sum_{n=m}^{\infty} \frac{|\chi_n \Im D_{n-1}(y)|}{n} \ll \sup_{\nu\in\mathbb{N}} |\chi_\nu| \sum_{n=m}^{\infty} \frac{1}{n} \min\left(n\|y\|, \frac{1}{n\|y\|}\right) = O\left(\min\left(1, \frac{1}{m\|y\|}\right)\right),$$

$$\sum_{n=m}^{\infty} |(\chi_n - \chi_{n+1}) \Im D_n(y)| \leq \sum_{k=0}^{\infty} \left(\max_{n\in[2^k m, 2^{k+1} m)} |\Im D_n(y)| \right) \mathrm{var}_{[2^k m, 2^{k+1} m)} \chi$$

$$\ll \sum_{k=0}^{\infty} \min\left(2^{k+1} m\|y\|, \frac{1}{2^k m\|y\|}\right) = O\left(\min\left(1, \frac{1}{m\|y\|}\right)\right).$$

To prove (2,iii), we write

$$\sum_{n=m}^{N_m} \frac{\sin 2\pi ny}{\pi n} = \frac{1}{2} - \mathrm{frac}(y) - \left(\sum_{n=1}^{m-1} + \sum_{n=N_m+1}^{\infty} \right) \frac{\sin 2\pi ny}{\pi n},$$

and for $|y| \le 1/2$, $N_m \ge M$ apply the following simple estimates

$$\frac{1}{2} - \mathrm{frac}(y) = \frac{\operatorname{sign} y}{2} + O(|y|), \quad \sum_{n=1}^{m-1} \frac{\sin 2\pi ny}{\pi n} = O(m|y|),$$
$$\sum_{n=N_m+1}^{\infty} \frac{\sin 2\pi ny}{\pi n} = O\left(\min\left(1, \frac{1}{M|y|}\right)\right).$$

The proof of (2,iv) is analogous:

$$\sum_{n=m}^{N_m} \frac{\cos 2\pi ny}{\pi n} = -\frac{1}{\pi} \ln|2\sin \pi y| - \left(\sum_{n=1}^{m-1} + \sum_{n=N_m+1}^{\infty} \right) \frac{\cos 2\pi ny}{\pi n},$$
$$-\frac{1}{\pi} \ln|2\sin \pi y| = \frac{1}{\pi} \ln \frac{1}{|y|} + O(1), \quad \sum_{n=1}^{m-1} \frac{\cos 2\pi ny}{\pi n} = \frac{1}{\pi} \ln m + O(1) + O(m^2y^2),$$
$$\sum_{n=N_m+1}^{\infty} \frac{\cos 2\pi ny}{\pi n} = O\left(\frac{1}{M|y|}\right), \quad |y| \le \frac{1}{2}.$$

In the proof of lemma 2, we will make use of the following properties of the partial quotients and the convergents, cf.[14], Chapter 10, or [28], Chapter 1:

$$x = \frac{a_j}{q_j} + \delta_j, \quad \frac{1}{2q_jq_{j+1}} < |\delta_j| \le \frac{1}{q_jq_{j+1}}, \quad \operatorname{sign} \delta_j = (-1)^j;$$
$$q_{j+1} = k_{j+1}q_j + q_{j-1}; \quad (a_j, q_j) = 1; \quad \frac{a_{j+1}}{q_{j+1}} - \frac{a_j}{q_j} = \frac{(-1)^j}{q_jq_{j+1}}, \; j = 0, 1, \ldots \tag{4}$$

where $a_0 := 0$.

Fix an integer $j \ge 0$, for brevity denote

$$\frac{a_j}{q_j} := \frac{a}{q}, \; k_{j+1} := K, \; q_{j+1} := Q, \; |\delta_j| := \delta, \quad A_{j,*}(\chi, x) := A_*,$$

and represent the sum A_* as follows:

$$A_* = B_* + C_*, \quad B_* := \sum_{m \in \mathcal{B}} \frac{\sigma_{m,*}(\chi, mx)}{m}, \quad C_* := \sum_{m \in \mathcal{C}} \frac{\sigma_{m,*}(\chi, mx)}{m}$$

where

$$\mathcal{B} := \left\{ m \in [q, Q), \quad \|mx\| \geq \frac{1}{8q} \right\}, \quad \mathcal{C} := \left\{ m \in [q, Q), \quad \|mx\| < \frac{1}{8q} \right\}.$$

Since $(a, q) = 1$, for each fixed $k \in \mathbb{N}$ the set of numbers $ma = (kq + l)a, \ l = 0, 1, \ldots, q - 1$ represents all residues mod$q$. Therefore,

$$\sum_{kq<m<(k+1)q} \left\| \frac{ma}{q} \right\|^{-1} = \sum_{0<l<q} \left\| \frac{l}{q} \right\|^{-1} = O(q \ln{(eq)}).$$

Moreover, if $q \leq m < Q$, then by (4) we also have

$$\left| mx - \frac{ma}{q} \right| \leq \frac{m}{qQ} \leq \frac{1}{q},$$

and since $m\|mx\| \geq 1$ for $m \in \mathcal{B}$, we obtain using (2,i):

$$B_* \ll \sum_{k=1}^{\infty} \sum_{m\in(kq,(k+1)q),\, \|mx\|\geq 1/(8q)} \frac{1}{m^2\|mx\|} \ll \sum_{k=1}^{\infty} \frac{q \ln{(eq)}}{k^2 q^2} \ll \frac{\ln{(eq)}}{q}. \tag{5}$$

Turn to the estimate of the sum C. Keeping in mind (see (4)) that δ satisfies the two-sided estimate $\frac{1}{2qQ} \leq \delta \leq \frac{1}{qQ}$, we see that the set $\mathcal{C}$ is contained in the union of the two following (finite, or possibly empty) progressions

$$\mathcal{C} \subset \mathcal{C}_1 \cup \mathcal{C}_2, \quad \mathcal{C}_1 := \left\{ m = kq, \ 1 \leq k \leq \frac{K}{4} \right\}, \quad \mathcal{C}_2 := \left\{ m = kq + l^*, \ \frac{K}{4} \leq k \leq K \right\},$$

l^* – the $\pmod q$-residue of the number $(-1)^{j+1}$, i.e. (cf. (4)) $l^* = q \operatorname{frac}\left(\frac{aQ}{q}\right)$; we have

$$\|mx\| = m\delta, \ m \in \mathcal{C}_1; \quad \|mx\| = \frac{1}{q} - m\delta, \ m \in \mathcal{C}_2.$$

We further subdivide the progression $\mathcal{C}_1$ into two parts:

$$\mathcal{C}_1' := \left\{ m = kq, \ 1 \leq k < \frac{\sqrt{K}}{2} \right\}, \quad \mathcal{C}_1'' := \left\{ m = kq, \ \frac{\sqrt{K}}{2} \leq k \leq \frac{K}{4} \right\}.$$

Applying the two-sided estimate of δ once again, we obtain

$$m\|mx\| = k^2 q^2 \delta \leq \frac{k^2 q}{Q} \leq 1, \quad m \in \mathcal{C}_1'; \quad m\|mx\| = k^2 q^2 \delta \geq \frac{k^2 q}{2Q} \geq \frac{1}{8}, \quad m \in \mathcal{C}_1'',$$

so that it follows by application of (2,i) that

$$\sigma_{m,*}(\chi, mx) = O\left(\frac{1}{m\|mx\|}\right) = O\left(\frac{1}{k^2q^2\delta}\right), \quad m \in \mathcal{C}_1'';$$
$$\|mx\| \geq \frac{1}{qQ}, \quad m\|mx\| \geq \frac{1}{q}, \quad \sigma_{m,*}(\chi, mx) = O(\ln eq),\ m \in \mathcal{C}_2.$$

From here we obtain the estimate similar to (5):

$$\sum_{m\in\mathcal{C}_1''\cup\mathcal{C}_2} \frac{|\sigma_{m,*}(\chi, mx)|}{m} \ll \sum_{k\geq\sqrt{K}/2} \frac{1}{k^3q^3\delta} + \sum_{K/4\leq k\leq K} \frac{\ln q}{kq} = O\left(\frac{\ln eq}{q}\right). \tag{6}$$

For the sums over the remaining progression $\mathcal{C}_1'$, the following estimates are valid, according to (2,i) and (2,ii) with $m = kq,\ \|mx\| = kq\delta,\ 1 \leq k \leq \sqrt{K}/2$:

$$\text{(i)} \sum_{m\in\mathcal{C}_1'} \frac{|\sigma_{m,*}(\chi, mx)|}{m} \ll \sum_{m\in\mathcal{C}_1'} \frac{1}{m} \ln \frac{1}{m\|mx\|} = \sum_{1\leq k\leq\sqrt{K}/2} \frac{1}{kq} \ln \frac{1}{(kq)^2\delta}$$
$$= \frac{1}{q}\left(\ln \frac{1}{q^2\delta} \sum_{1\leq k\leq\sqrt{K}/2} \frac{1}{k} - 2 \sum_{1\leq k\leq\sqrt{K}/2} \frac{\ln k}{k}\right) = \frac{\ln^2 Q}{4q} + O\left(\frac{\ln^2 q + \ln Q}{q}\right),$$
$$\text{(ii)} \sum_{m\in\mathcal{C}_1'} \frac{|\Im\sigma_{m,*}(\chi, mx)|}{m} \ll \sum_{m\in\mathcal{C}_1'} \frac{1}{m} = \sum_{1\leq k\leq\sqrt{K}/2} \frac{1}{kq} = \frac{\ln eQ}{2q} + O\left(\frac{\ln eq}{q}\right). \tag{7}$$

Summarizing the estimates (5) – (7), we see that

$$A_* = O\left(\frac{\ln^2 eQ}{q}\right), \quad \Im A_* = O\left(\frac{\ln eQ}{q}\right).$$

This completes the proof of (3,i) and (3,ii), because $q = q_j,\ Q = q_{j+1}$.

To prove the asymptotic relations (3,iii), we first take care of the "remainder terms" in (2,iii) and (2,iv) for $m \in \mathcal{C}_1' \cap [1, M]$ and $y = \|mx\|$, where again we keep in mind that

$q = q_j,\ Q = q_{j+1},\ K = k_{j+1}$, see (4):

$$
\begin{aligned}
&\text{(i)} \sum_{m\in\mathcal{C}_1'\cap[1,M]} \frac{1}{m}\left(m\|mx\| + \min\left(1, \frac{1}{M\|mx\|}\right)\right)\\
&= \sum_{1\le k\le \min(\sqrt{K}/2, M/q)} \left(kq\delta + \frac{1}{kq}\min\left(1,\frac{1}{Mkq\delta}\right)\right) \ll \frac{1}{q}\left(1 + \sum_{1\le k\le \min(M/q, 2Q/M)} \frac{1}{k}\right)\\
&= O\left(\frac{1}{q}\left(1+\ln e\min\left(\frac{M}{q},\frac{Q}{M}\right)\right)\right),\\
&\text{(ii)} \sum_{m\in\mathcal{C}_1'\cap[1,M]} \frac{1}{m}\left(1 + (m\|mx\|)^2 + \frac{1}{M\|mx\|}\right) = O\left(\frac{1}{q}\left(\ln e\min\left(\frac{M}{q}, Q\right) + \frac{Q}{M}\right)\right). \quad (8)
\end{aligned}
$$

If $j = 0, 1, \ldots, J-1$, then by the definition of the number $M = M(\Omega)$ we have $q_{j+1} \le M$, so that the above quantities are incorporated into the remainder terms of the relations (3,iii). We also have

$$
C'_{1,j} = \left\{m:\ m = kq_j,\ 1\le k\le \frac{\sqrt{k_{j+1}}}{2}\right\} \subset [1, M], \quad j = 0, 1, \ldots, J-1,
$$

and it is easy to see that the same conclusions remain true for $j = J$, if we additionally assume that $M(\Omega) \ge \frac{q_{J+1}}{2}$.

Thus, summarizing (2, iii) and (2, iv) over the progression $C'_{1,j} = C'_{1,j}\cap[1,M]$, and making use of (4) and the estimates (5)– (7), we obtain that for all $j = 0, 1, \ldots, J$

$$
\begin{aligned}
&\Im A_j(\Omega, x) = \sum_{m\in C'_{1,j}} \frac{\Im\sigma_m(\Omega, mx)}{m} + O\left(\frac{\ln eq_j}{q_j}\right)\\
&= \operatorname{sign}\delta_j \sum_{1\le k\le\sqrt{k_{j+1}}/2} \frac{1}{kq_j} + O\left(\frac{\ln eq_j}{q_j}\right) = \frac{(-1)^j \ln q_{j+1}}{2q_j} + O\left(\frac{\ln eq_j}{q_j}\right),\\
&\Re A_j(\Omega, x) = \sum_{m\in C'_{1,j}} \frac{\Re\sigma_m(\Omega, mx)}{m} + O\left(\frac{\ln q_{j+1} + \ln^2 q_j}{q_j}\right) = \frac{2}{\pi}\sum_{1\le k\le\sqrt{k_{j+1}}/2} \frac{1}{kq_j}\ln\frac{1}{(kq_j)^2|\delta_j|}\\
&+O\left(\frac{\ln q_{j+1} + \ln^2 q_j}{q_j}\right) = \frac{\ln^2 q_{j+1}}{2\pi q_j} + O\left(\frac{\ln q_{j+1} + \ln^2 q_j}{q_j}\right)
\end{aligned}
$$

(for $j = J$ we also made use of the additional assumption $M(\Omega) \ge \frac{q_{J+1}}{2}$). The relations (3,iii) follow, and the proof of lemma 2 is complete.

Let us finish the proof of theorem 1. Let us first keep our original assumption, that x is an irrational number, and consider an expanding sequence of $\mho$-domains $\{\Omega_r\}$. Recall the representation

$$U_\Omega(\chi,x)=\sum_{j=0}^{J}A_j(\chi,\Omega_r,x), \tag{9}$$

for $\Omega=\Omega_r,\ J=J(\Omega_r,x)$. Then $J(\Omega_r,x)\to\infty,\ r\to\infty$, and according to lemma 1 and lemma 2, for each fixed j the following limits exist, and satisfy the indicated estimates:

$$\begin{aligned}
&A_{j\infty}(\chi,x):=\lim_{r\to\infty}A_j(\chi,\Omega_r,x)=\sum_{m\in[q_j,q_{j+1})}\frac{\sigma_{m,\infty}(\chi,mx)}{m}=O\left(\frac{\ln^2 q_{j+1}}{q_j}\right),\\
&\Im A_{j\infty}(\chi,x)=O\left(\frac{\ln q_{j+1}}{q_j}\right);\quad \varepsilon_{j\infty}(x):=A_{j\infty}(x)-\left(\frac{\ln^2 q_{j+1}}{2\pi q_j}+i\frac{(-1)^j\ln q_{j+1}}{2q_j}\right),\\
&\Im\varepsilon_{j\infty}(x)=O\left(\frac{\ln q_j}{q_j}\right),\quad \Re\varepsilon_{j\infty}(x)=O\left(\frac{\ln q_{j+1}+\ln^2 q_j}{q_j}\right).
\end{aligned}\tag{10}$$

The denominators $\{q_j(x)\}$ grow at least exponentially, so that

$$\sum_{j=0}^{\infty}\frac{\ln^2 q_j(x)}{q_j(x)}\ll 1,\quad \sum_{j=0}^{\infty}\frac{\ln q_j(x)}{q_j(x)}\ll 1\,. \tag{11}$$

From here, applying also the estimates of the bounds A_{j*} in lemma 2, we infer that for an irrational x:
if $\Re\Xi(x)$ converges, then $U(\chi,x)$ is $\mho$-summabale, and $|U(\chi,x)|\ll\Re\Xi(x)$;
if $\Upsilon(x)$ converges, then $\Im U(\chi,x)$ is $\mho$-summabale, and $|\Im U(\chi,x)|\ll\Upsilon(x)$;
if $\Im\Xi(x)$ converges, then $\Im U(x)$ is $\mho$-summable, and $|\Im(U(x)-\Xi(x))|\ll 1$;
if $\Re\Xi(x)$ converges, then $\Re U(x)$ is $\mho$-summable, and $|\Re(U(x)-\Xi(x))|\ll\Upsilon(x)$.

Now consider a full sequence of $\mho$-domains $\{\Omega_r\}$. Then for an irrational x and each sufficiently large J one can find such $r=r(J)$ that $q_{J+1}>M(\Omega_r)\geq\frac{q_{J+1}}{2}$. There are two possibilities: either a) $M(\Omega_{r(J)})<q_J$, or b) $M(\Omega_{r(J)})\geq q_J$. In the case a) we have $M(\Omega_{r(J)})\geq\frac{q_{J+1}}{2}\geq q_{J-1}$, and $q_{J+1}\leq 2q_J$, so that $q_{J-1}\leq M(\Omega_{r(J)})<q_J$,

$$U_{\Omega_{r(J)}}(x)=\sum_{j=0}^{J-1}A_j(\Omega_{r(J)},x)$$

where the estimates (3,iii) are true for all $j=0,1,\ldots,J-1$. In addition, in this case we obviously have

$$\frac{\ln^2 q_{J+1}}{2\pi q_J}+i\frac{(-1)^j\ln q_{J+1}}{2q_J}=O\left(\frac{\ln^2 q_J}{q_J}\right).$$

In the case b) we have $q_J \le M(\Omega_{r(J)}) < q_{J+1}$,

$$U_{\Omega_{r(J)}}(x) = \sum_{j=0}^{J} A_j(\Omega_{r(J)}, x),$$

where the estimates (3,iii) hold for all $j = 0, 1, \ldots, J$, because $M(\Omega_r) \ge \frac{q_{J+1}}{2}$. Therefore, in each case we have

$$U_{\Omega_{r(J)}}(x) = \sum_{j=0}^{J} \left(\frac{\ln^2 q_{j+1}}{2\pi q_j} + i\frac{(-1)^j \ln q_{j+1}}{2q_j} + \varepsilon_j(\Omega_{r(J)}, x) \right),$$

$$\Re\varepsilon_j(\Omega_{r(J)}, x) = O\left(\frac{\ln^2 q_j + \ln q_{j+1}}{q_j} \right), \quad \Im\varepsilon_j(\Omega_{r(J)}, x) = O\left(\frac{\ln q_j}{q_j} \right), \quad j = 0, 1, \ldots, J.$$

Thus, for each full sequence of $\mho$-domains $\{\Omega_r\}$, the following implications are true:
if $\Im\Xi(x)$ diverges, then the sequence $\{\Im U_{\Omega_r}(x)\}$ is divergent;
if $\Re\Xi(x)$ diverges, then the sequence $\{\Re U_{\Omega_r}(x)\}$ is divergent.

To finish the proof of theorem 1, we need to consider the remaining case of a rational $x = \frac{a}{q}$, $(a, q) = 1$. In this case, the continued fraction (1) is finite, and there exists a finite $J = J(x)$ ($J =$ the smallest number of "floors" in the representation (1)) such that if $M = M(\Omega) \ge q_J = q$, then

$$U_\Omega(\chi, x) = \sum_{j=0}^{J} A_j(\chi, \Omega, x).$$

It is easy to see that for $j < J$ the estimates of lemma 2 remain true; the relations (5), (6) remain valid also for $j = J$, and a modification is needed only in the consideration of the sums over the progression

$$\mathcal{C}'_{1J}(\Omega) = \left\{ m = kq,\ 1 \le k \le \frac{M}{q} \right\},$$

see (7), where $\|mx\| = 0$. Here, we obviously have

$$\Im\sigma_m(\chi, mx) = 0, \quad \Re\sigma_m(\chi, mx) = \sum_{n=m}^{N_m} \frac{\chi_{m,n}}{\pi n} + \sum_{n=m+1}^{M_m} \frac{\chi_{n,m}}{\pi n},$$

and consequently, in the particular case of $\chi \equiv 1$,

$$\Re\sigma_m(mx) = \sum_{n=m}^{N_m} \frac{\chi_{m,n}}{n} + \sum_{n=m+1}^{M_m} \frac{\chi_{n,m}}{\pi n} \ge \frac{2}{\pi} \ln\frac{M}{m} + O\left(\frac{1}{m}\right),$$

so that

$$\Re A_J(\Omega,x) \geq \frac{2}{\pi} \sum_{1\leq k\leq M/q} \frac{1}{kq} \ln \frac{M}{kq} + O\left(\frac{\ln eq}{q}\right) = \frac{1}{\pi q}\left(\ln^2 \frac{M}{q} + O(\ln M)\right) \to \infty, \quad M \to \infty.$$

Thus, for $x \in \mathbb{Q}$ the sin-series $\Im U\chi, x)$ with a general real CS-sequence χ is $\mho$-summable; for $\chi \equiv 1$, the cos-series $\Re U(x)$ is $\mho$-divergent, and moreover,

$$\lim_{r\to\infty} \Re U_{\Omega_r}(x) = +\infty$$

for each expanding sequence of $\mho$ domains $\{\Omega_r\}$. This completes the proof of theorem 1.

Proof of Theorem 2. For a collection of natural numbers $\mathbf{k} = (k_1, \dots, k_{j-1}, k_j) \in \mathbb{N}^j$, let (see (1))

$$[k_1, \dots, k_{j-1}] = \frac{a_{j-1}}{q_{j-1}}$$

and consider an interval $\omega = \omega(\mathbf{k})$ with the end-points

$$[k_1, \dots, k_{j-1}, k_j] = \frac{a_j}{q_j}, \quad [k_1, \dots, k_{j-1}, k_j + 1] = \frac{a'_j}{q'_j}\,. \tag{12}$$

By the basic property of the continued fraction, see (4), we have

$$\left|\frac{a_{j-1}}{q_{j-1}} - \frac{a_j}{q_j}\right| = \frac{1}{q_j q_{j-1}}\,,$$

and further,

$$\text{(i) } q_j(x) = q_j,\ x \in \omega; \quad \text{(ii) } |\omega| = \left|\frac{a_j}{q_j} - \frac{a'_j}{q'_j}\right| = \frac{1}{q_j q'_j}\,, \quad \text{(iii)}^1 \bigcup_{\mathbf{k}\in\mathbb{N}^j} \omega(\mathbf{k}) = (0,1)\,. \tag{13}$$

Consider the partitioning of ω by the points

$$\frac{a_{j+1,k}}{q_{j+1,k}} = [k_1, \dots, k_{j-1}, k_j, k] = \frac{a_j k + a_{j-1}}{q_j k + q_{j-1}}, \quad k = 1, 2, \dots$$

Denote ω_k the sub-interval of ω whose endpoints are two consecutive fractions $\frac{a_{j+1,k}}{q_{j+1,k}}$ and $\frac{a_{j+1,k+1}}{q_{j+1,k+1}}$. Then

$$\text{(i) } \varphi_j(x) = \frac{\ln q_{j+1,k}}{q_j} = \frac{\ln(q_j k + q_{j-1})}{q_j}, \quad x \in \omega_k;$$

$$\text{(ii) } |\omega_k| = \left|\frac{a_{j+1,k}}{q_{j+1,k}} - \frac{a_{j+1,k+1}}{q_{j+1,k+1}}\right| = \frac{1}{q_{j+1,k}q_{j+1,k+1}}\,, \quad \text{(iii) } \bigcup_{k=1}^{\infty} \omega_k = \omega\,. \tag{14}$$

[1]Here and in the similar partitions below, we disregard all rational points

Let

$$\mathcal{E}_\omega(z) := \{x: \ x \in \omega, \ \varphi_j(x) > z\}, \quad \mu(\omega, z) := \operatorname{meas} \mathcal{E}_\omega(z). \tag{15}$$

Then using (14) and (13, ii) we conclude that

$$\mathcal{E}_\omega(z) \subset \bigcup_{k:\ q_{j+1,k} > e^{q_j z}} \omega_k, \quad \mu(\omega, z) = \sum_{k:\ q_{j+1,k} > e^{q_j z}} |\omega_k| = \sum_{k:\ q_{j+1,k} > e^{q_j z}} \left| \frac{a_{j+1,k}}{q_{j+1,k}} - \frac{a_{j+1,k+1}}{q_{j+1,k+1}} \right|$$

$$\leq \frac{1}{e^{q_j z} q_j} = q_j' e^{-q_j z} |\omega| \leq 2 q_j e^{-q_j z} |\omega|; \quad \mu(\omega, z) \leq \min\left(1, \, 2 q_j e^{-q_j z}\right) |\omega| \, . \tag{16}$$

Since $q_j \geq F_j$ and

$$\{x: \ x \in (0,1), \ \varphi_j > z\} = \bigcup_{\mathbf{k} \in \mathbb{N}^j} \mathcal{E}_{\omega(\mathbf{k})}(z)$$

it follows that

$$\begin{aligned} &\mu(\omega, z) \leq \min\left(1, \, 2 F_j e^{-F_j z}\right) |\omega| \, , \\ &\mu_j(z) \leq \min\left(1, \, 2 F_j e^{-F_j z}\right) \sum_{\mathbf{k} \in \mathbb{N}^j} |\omega(\mathbf{k})| = \min\left(1, \, 2 F_j e^{-F_j z}\right) \, . \end{aligned} \tag{17}$$

In particular, φ_j is exponentially integrable, and for $\lambda < F_j$

$$\begin{aligned} &\int_0^1 e^{\lambda \varphi_j(x)} \, dx = -\int_0^\infty e^{\lambda z} \, d\mu_j(z) = 1 + \lambda \int_0^\infty e^{\lambda z} \, \mu_j(z) \, dz \\ &\leq 1 + \lambda \int_0^\infty e^{\lambda z} \, \min\left(1, \, 2 F_j e^{-F_j z}\right) = e^{\lambda \xi_j} \left(1 + \frac{1}{F_j - \lambda}\right), \quad \xi_j := \frac{\ln 2 F_j}{F_j}. \end{aligned} \tag{18}$$

For a fixed j, let us consider the sequence of functions

$$\Upsilon_{j,l} := \sum_{\nu=j}^{l} \varphi_\nu, \quad l = j, j+1, \ldots$$

On each interval $\omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{N}^l$ all functions φ_ν with $\nu < l$ are constant, so that the sum $\Upsilon_{j,l-1}$ is also constant. Thus, applying (18) and (16) and induction in l with $l \geq j$, for $\lambda < F_j$ we obtain the estimate

$$\int_0^1 e^{\lambda \Upsilon_{j,l}(x)} \, dx \leq \prod_{\nu=j}^{l} e^{\lambda \xi_\nu} \left(1 + \frac{1}{F_\nu - \lambda}\right).$$

Letting $l \to \infty$, from here we derive the claims A) of theorem 2.

To prove the claim B), for a given positive number ε and $j = 0, 1, \ldots$ let us consider the sets

$$\mathcal{F}_j(\varepsilon) := \left\{ x : \ x \in (0,1), \ \varphi_j(x) > z_j, \quad z_j := \frac{1}{(j+1)^2 \varepsilon} \right\}, \quad \mathcal{F}(\varepsilon) := \bigcup_{j=0}^{\infty} \mathcal{F}_j(\varepsilon).$$

According to (15), (16), for each interval $\omega = \omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{N}^j$, the part $\mathcal{E}_\omega(z_j)$ of $\mathcal{F}_j(\varepsilon)$ is an interval $I = I_{j,\varepsilon}(\mathbf{k})$ of the length

$$|I| \leq \frac{1}{q_j} \exp\left(-\frac{q_j}{(j+1)^2 \varepsilon} \right),$$

and keeping in mind that $\alpha > 2$, $|\omega| \geq q_j^{-2}$, we have

$$\mathcal{E}_\omega(z_j) = I, \quad \left(\ln \frac{1}{I} \right)^{-\alpha} \leq \left(\ln q_j + \frac{q_j}{(j+1)^2 \varepsilon} \right)^{-\alpha} \leq \frac{\varepsilon^\alpha (j+1)^{2\alpha}}{q_j^\alpha} \leq \frac{\varepsilon^\alpha (j+1)^{2\alpha}}{F_j^{\alpha-2}} |\omega|.$$

Therefore,

$$\text{(i) } \mathcal{F}_j(\varepsilon) = \bigcup_{\mathbf{k} \in \mathbb{N}^j} I_{j,\varepsilon}(\mathbf{k}); \quad \text{(ii) } \sum_{\mathbf{k} \in \mathbb{N}^j} \left(\ln \frac{1}{|I_{j,\varepsilon}(\mathbf{k})|} \right)^{-\alpha} \leq \frac{\varepsilon^\alpha (j+1)^{2\alpha}}{F_j^{\alpha-2}} \sum_{\mathbf{k} \in \mathbb{N}^j} |\omega(\mathbf{k})| = \frac{\varepsilon^\alpha (j+1)^{2\alpha}}{F_j^{\alpha-2}}.$$

Consequently, the set $\mathcal{F}(\varepsilon)$ is covered by the collection of intervals

$$\mathcal{I}(\alpha, \varepsilon) := \left\{ \{ I_{j,\varepsilon}(\mathbf{k}) \}_{\mathbf{k} \in \mathbb{N}^j} \right\}_{j=0}^{\infty},$$

and

$$\sum_{I \in \mathcal{I}(\alpha,\varepsilon)} \left(\ln \frac{1}{|I|} \right)^{-\alpha} \leq \varepsilon^\alpha \sum_{j=0}^{\infty} \frac{(j+1)^{2\alpha}}{F_j^{\alpha-2}} \overset{\alpha}{\ll} \varepsilon^\alpha, \quad \alpha > 2.$$

In the complement of the set $\mathcal{F}(\varepsilon)$ the series Υ converges:

$$\Upsilon(x) = \sum_{j=0}^{\infty} \varphi_j(x) \leq \sum_{j=0}^{\infty} \frac{1}{(j+1)^2 \varepsilon} < \infty,$$

which completes the proof of the claim B) of theorem 2.

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