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I dedicate this paper to my teacher Sergei Alexandrovich Telyakovskii on occasion of his 70-th birthday.

For a natural d, denote $\mathbb{R}^d$ the d-dimensional real Euclidean space of vectors $\mathbf{v} = (v_1, \ldots, v_d)$; $\mathbb{Z}^d$ -the integral lattice in $\mathbb{R}^d$; $\mathbb{N}^d$ - the subset of vectors $\mathbf{n} \in \mathbb{R}^d$ with the natural coordinates; $\mathbb{Z}^d/\mathbb{N}$ - the set of vectors in $\mathbb{R}^d$ with the rational coordinates.

A set $\omega \subset \mathbb{R}^d$ is called *coordinate-wise connected* iff the intersection of ω with every line parallel to one of the coordinate axes is either an interval, or an empty set. Denote Ω^d the class of all coordinate-wise connected domains in $\mathbb{R}^d$.

S.A. Telyakovskii [6], [7] established the following remarkable result concerning multiple sin-sums with the linear phase.

Theorem 1 *For $d \geq 2$*

$$\sup_{\omega\in\Omega^d}\sup_{\mathbf{x}\in\mathbb{R}^d}\left|\sum_{\mathbf{n}\in\omega\cap\mathbb{N}^d}\frac{\sin n_1x_1}{n_1}\cdots\frac{\sin n_dx_d}{n_d}\right| < \infty.$$

This result has found noteworthy applications in estimates of Kolmogorov's widths for classes of functions with the bounded mixed derivative, and in general, in the approximation of functions by "hyperbolic cross". For some later developments, we refer to the paper [8], and to the literature cited therein.

The following theorem was proved in [1], on the base of I.M. Vinogradov's method of exponential sums [16]. It concerns 1-d discrete oscillatory Hilbert transforms with the polynomial phase of higher degree (cf. also [13]). We use the notation $\mathcal{P}^r,\ r \in \mathbb{N}$ for the set of univariate algebraic polynomials p of degree r with the real coefficients, such that $p(0) = 0$.

Theorem 2 *For a polynomial $p \in \mathcal{P}^r$, let*

$$h_m(p) := \sum_{n=1}^{m}\frac{e^{ip(n)} - e^{ip(-n)}}{n}\ , \quad m \geq 1.$$

Then

$$\sup_{p\in\mathcal{P}^r}\sup_m |h_m(p)| < \infty, \tag{1}$$

and for each fixed polynomial $p\in\mathcal{P}^r$*, there exists the limit* $h(p) := \lim_{m\to\infty} h_m(p)$*.*

Here we prove theorem 3, that unifies and somewhat generalizes these two results. It concerns the multiple discrete Hilbert transforms with the polynomial phase.

A numerical sequence $f = \{f_n\}_{n\in\mathbb{N}}$ satisfies the Littlewood – Paley condition (see [18], Chapter 15), iff $\|f\|_\infty := \sup_n |f_n| < \infty$, and

$$A(f) := \sup_n \sum_{n\le m\le 2n} |f_m - f_{m+1}| < \infty.$$

In the sequel, we call such sequences *slow*; let us use the notation $\mathcal{S}$ for the class of all slow sequences, $\|f\|_{\mathcal{S}} := \|f\|_\infty + A(f)$ for the norm in $\mathcal{S}$. Clearly, if a sequence f has a bounded variation in the usual sense, i. e. $f\in \mathrm{BV}$, then this sequence is slow. However, the class $\mathcal{S}$ is essentially wider than BV. For example, for each fixed real $t\neq 0$ we have $f_t := \{n^{it}\} = \left\{e^{it\ln n}\right\}_{n\in\mathbb{N}} \in \mathcal{S}$, but $f_t\notin \mathrm{BV}$.

We will say that d-indexed sequence $f = \{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ is *coordinate wise slow* (notation: $f\in\mathcal{S}^d$), iff restrictions of f onto the lines parallel with coordinate axes are uniformly slow:

$$\|f\|_{\mathcal{S}^d} := \|f\|_\infty + \max_{1\le k\le d}\sup_{\mathbf{n}_k}\sup_n \sum_{n\le m\le 2n} |f_{\mathbf{n}_k+m\mathbf{e}_k} - f_{\mathbf{n}_k+(m+1)\mathbf{e}_k}| < \infty,$$

where $\mathbf{e}_k := (0,\dots,0,\overset{k}{1},0,\dots,0)$, $k=1,\dots,d$ denotes the standard basis in $\mathbb{R}^d$, and for $\mathbf{n} = (n_1,\dots,n_k,\dots,n_d)$, $\mathbf{n}_k := \mathbf{n} - n_k\mathbf{e}_k$. Obviously, the characteristic function of a coordinate-wise connected domain, i. e. $\omega\in\Omega^d$, is slow.

We will also use the following notations: $\mathcal{P}^{r,d}$ the set of collections $\vec{p} = (p_1,\dots,p_d)$, where $p_k\in\mathcal{P}^r$; $\square_{\mathbf{m}}$ – the parallelepiped $\{\mathbf{n}\in\mathbb{N}^d :\ n_k\le m_k,\ k=1,\dots,d\}$. The symbols $\overset{r}{\ll}$, $\overset{r,d}{\ll}$ in relations of the type $A\overset{r}{\ll} B$, $A\overset{r,d}{\ll} B$ mean that that there are (finite) factors c_r, $c_{r,d}$ that depend only on the indicated parameters r, or r,d, such that $|A|\le c_r|B|$, or respectively, $|A|\le c_{r,d}|B|$.

Theorem 3 *For* $\vec{p}\in\mathcal{P}^{r,d}$, $\mathbf{m}\in\mathbb{N}^d$*, let*

$$h_{\mathbf{m}}(f,\vec{p}) := \sum_{\mathbf{n}\in\square_{\mathbf{m}}} f_{\mathbf{n}} \prod_{k=1}^d \frac{e^{ip_k(n_k)} - e^{ip_k(-n_k)}}{n_k}.$$

If $f\in\mathcal{S}^d$*, then*

$$\sup_{\mathbf{m}\in\mathbb{N}^d}\sup_{\vec{p}} |h_{\mathbf{m}}(f,\vec{p})| \overset{r,d}{\ll} \|f\|_{\mathcal{S}^d} \tag{2}$$

and for each collection of polynomials $\vec{p}\in\mathcal{P}^{r,d}$*, the limit* $h(f,\vec{p}) := \lim_{\min_k\to\infty} h_{\mathbf{m}}(f,\vec{p})$ *exists.*

It is interesting to compare this result with the recent one of M.Z. Garaev [4], who considered the sequence of partial sums

$$h_N(x) := \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{\sin mnx}{mn}, \quad N = 1, 2, \ldots$$

Garaev proved that there exist real numbers x for which the sequence $h_N(x)$ *diverges* as $N \to \infty$.
Proof of the theorem. For $p \in \mathcal{P}^r$, denote

$$e_-^{ip(t)} := \frac{e^{ip(t)} - e^{ip(-t)}}{2i}$$

and for $n \in \mathbb{N}$, $\mathbf{n} \in \mathbb{N}^d$, $\vec{p} \in \mathcal{P}^{r,d}$, introduce the following exponential sums and integrals:

$$\begin{aligned}
&T_n(p) := \frac{1}{n} \sum_{m=1}^{n} e_-^{ip(n)}, \quad I_n(p) := \frac{1}{n} \int_0^n e_-^{ip(t)}\, dt, \quad T_0(p) = I_0(p) := 0; \\
&A_n(p) := T_n(p) - T_{n-1}(p), \quad B_n(p) := B_n(p) - B_{n-1}(p); \\
&A_{\mathbf{n}}(\vec{p}) := \prod_{k=1}^{d} A_{n_k}(p_k)\,, \quad B_{\mathbf{n}}(\vec{p}) := \prod_{k=1}^{d} B_{n_k}(p_k)\,.
\end{aligned}$$

We have

$$\frac{e_-^{ip(n)}}{n} = A_n(p) + \frac{T_{n-1}(p)}{n}\,, \quad \int_{n-1}^{n} \frac{e_-^{ip(t)}}{t}\, dt = B_n(p) + \frac{I_{n-1}(p)}{n}\,. \tag{3}$$

It was proved in [1] that

$$a)\ \sup_{p \in \mathcal{P}^r} \sum_{n \in \mathbb{N}} \frac{|T_{n-1}(p)|}{n} < \infty, \quad b)\ \sup_{p \in \mathcal{P}^r} \sum_{n \in \mathbb{N}} \frac{|I_{n-1}(p)|}{n} < \infty. \tag{4}$$

The claim b) in these relations concerns the integral Hilbert transform with the polynomial phase

$$g(p) := \lim_{m \to \infty} g_m(p), \quad g_m(p) := \int_0^m \frac{e_-^{ip(t)}}{t}\, dt.$$

The global boundedness $\sup_{m>0} \sup_{p \in \mathcal{P}^r} |g_m(p)| < \infty$, and the existence of the limit $g(p)$ are the results due to E. Stein and S. Wainger [15].

The claim a) in (4) is of a somewhat more complicated nature, because the arithmetical properties of the coefficient vector of the polynomial p play a significant role, and the proof requires the application of the circular method of G.H. Hardy – J.E. Littlewood – I.M. Vinogradov. This claim was proved in [1]. Independently and somewhat later than in [1], the global boundedness result (1) was established by Stein and Wainger, see [13] and also [14], p. 373.

For a subset of indices $\mathcal{A} \subset [1,d]$ (possibly empty), let us consider the composite integral-discrete products

$$C_{\mathbf{n}}(\mathcal{A},\vec{p}) := \left(\prod_{k\in\mathcal{A}} A_{n_k}(p_k)\right)\left(\prod_{k\in[1,d]\setminus\mathcal{A}} B_{n_k}(p_k)\right), \quad \mathbf{n}\in\mathbb{N}^d,\ \vec{p}\in\mathcal{P}^{r,d}.$$

Then in view of (3) and (4), theorem 3 is a corollary, corresponding to the particular case of $\mathcal{A}=[1,d]$, of the following statement.

Theorem 4 *Assume that $f := \{f_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ is a coordinate-wise slow sequence, with $\|f\|_{\mathcal{S}^d} \le 1$, $\mathcal{A}\subset[1,d]$ – a subset of indices, and $\vec{p}\in\mathcal{P}^{r,d}$. Then the limit*

$$S(f,\mathcal{A},\vec{p}) := \lim_{\min_k m_k\to\infty} \sum_{\mathbf{n}\in\square_m} f_{\mathbf{n}} C_{\mathbf{n}}(\mathcal{A},\vec{p})$$

exists, and

$$|S(f,\mathcal{A},\vec{p})| \overset{r,d}{\ll} 1. \tag{5}$$

For a polynomial $p\in\mathcal{P}^r$, denote $\mathbf{x}=(x_1,\dots,x_r)$ the coefficients vector of $\frac{p}{2\pi}$, and let

$$p(t) = 2\pi P(\mathbf{x},t) := 2\pi(x_1 t + \cdots + x_r t^r), \quad p_*(t) := 2\pi(|x_1|t + \cdots + |x_r|t^r).$$

The following estimates are trivial

$$\begin{aligned} &\max(|T_n(p)|,|I_n(p)|) \le \min(1,p_*(n)); \\ &|C_n(p)| \le \max(|A_n(p)|,|B_n(p)|) \ll \frac{\min(1,p_*(n))}{n}, \end{aligned} \tag{6}$$

simply because $\left|e_-^{ip}\right| = |\sin p_-|$ where $p_-(t) := (p(t)-p(-t))/2$, and $|p_-| \le p_*$.

We split the proof of theorem 4 into the the consideration of the following 4 cases.

Case 1. $\mathcal{A}=\emptyset,\ d=1$.
Case 2. $\mathcal{A}=\emptyset,\ d\ge 2$.
Case 3. $\mathcal{A}=\{1\},\ d=1$.
Case 4. A general $\mathcal{A}\subset[1,d],\ d\ge 2$.

Case 1 is the simplest. Here we need to study the sum of integrals

$$S(f,p) = S(f,\emptyset,p) := \sum_{n\in\mathbb{N}} f_n B_n(p).$$

By Abel's transformation, for the m-th partial sum of this series we have

$$\begin{aligned} S_m(f,p) &:= \sum_{n=1}^{m} f_n B_n(p) = \sum_{n=1}^{m} f_n (I_n(p) - I_{n-1}(p)) \\ &= f_m I_m(p) + \sum_{n=1}^{m-1} I_n(p)\Delta f_n, \quad \Delta f_n := f_n - f_{n+1}. \end{aligned}$$

Therefore, it suffices to prove that $I_n(p) \to 0,\ n \to \infty$, and if $f \in \mathcal{S},\ \|f\|_{\mathcal{S}} \le 1$ then

$$\sum_{n=1}^{\infty} |I_n(p)\Delta f_n| \overset{r}{\ll} 1. \tag{7}$$

It follows from the definition of slow sequences that

$$\sum_{n=1}^{m} p_*(n)|\Delta f_n| \ll p_*(m), \qquad \sum_{n=m+1}^{\infty} {p_*}^{-\rho}(n)|\Delta f_n| \overset{\rho}{\ll} {p_*}^{-\rho}(m), \quad p \in \mathcal{P}^r, \quad \rho > 0. \tag{8}$$

Furthermore, for the integral I_n the following estimate is true (see also [1], [10])

$$|I_n(p)| \ll \min\left(p_*(n),\, {p_*}^{-\rho}(n)\right), \quad \rho := \frac{1}{r}. \tag{9}$$

In view of (6), this estimate is a corollary of the inequality

$$\left|\frac{1}{n}\int_0^n e^{ip(t)}\,dt\right| \ll \min\left(1,\, {p_*}^{-\rho}(n)\right), \quad p \in \mathcal{P}^r,$$

which is equivalent to Vinogradov's estimate of the "standard" oscillatory integral with the polynomial phase (see [16], Ch. 2, Lemma 4):

$$\left|\int_0^1 e^{ip(t)}\,dt\right| = \left|\int_0^1 e^{2\pi i P(\mathbf{x},t)}\,dt\right| \le \min\left(1,\, 32\min_k |x_k|^{-\rho}\right) \le \min\left(1,\, 32 r^{\rho} {p_*}^{-\rho}(1)\right).$$

Therefore, it follows from (9) that indeed, in the non-trivial cases, when p is not identically 0, $I_n(p) \to 0,\ n \to \infty$, and

$$\sum_{n=1}^{\infty} |I_n(p)\Delta f_n| \ll \sum_{n=1}^{\infty} \min\left(p_*(n), {p_*}^{-\rho}(n)\right)|\Delta f_n| \overset{r}{\ll} \min_m\left(p_*(m) + {p_*}^{-\rho}(m)\right) \overset{r}{\ll} 1.$$

Moreover, we have

$$\begin{aligned}|S(f,p) - S_m(f,p)| &= \left|\sum_{n=m}^{\infty} f_n B_n(p)\right| \le |f_m I_m(p)| + \sum_{n=m}^{\infty} |I_n(p)\Delta f_n| \\ &\ll \sum_{n=m}^{\infty} \min\left(p_*(n), {p_*}^{-\rho}(n)\right)|\Delta f_n| \overset{r}{\ll} \min\left(1, {p_*}^{-\rho}(m)\right),\end{aligned} \tag{10}$$

which provides the estimate of the rate of convergence of the series in the case 1, and concludes the proof of this case.

Case 2. Here we need to consider the multiple sum of integrals

$$S(f,\vec{p}) = S(f,\emptyset,\vec{p}) = \sum_{\mathbf{n}\in\mathbb{N}^d} f_{\mathbf{n}} B_{\mathbf{n}}(\vec{p}).$$

For this purpose, we apply a modification of Telyakovskii's central idea of the proof of his Theorem 1. Given a collection of polynomials $\vec{p} = (p_1, \dots, p_d) \in \mathcal{P}^{r,d}$, let us subdivide the domain of summation $\mathbb{N}^d$ into $d!$ sub-domains, according to the size of the polynomials $p_{*1}, \dots, p_{*d}$. Typical sub-domains are the algebraic octants

$$\omega = \omega(\vec{p}) := \left\{(n_1, \dots, n_d) \in \mathbb{N}^d : \ p_{*1}(n_1) \geq \dots \geq p_{*d}(n_d)\right\},$$
$$\omega_m = \omega_m(\vec{p}) := \left\{(n_1, \dots, n_d) \in \mathbb{N}^d : \ p_{*1}(n_1) \geq \dots \geq p_{*d}(n_d), \ n_d \geq m\right\}, \ m \in \mathbb{N}. \tag{11}$$

The other $d! - 1$ octants are obtained from ω by taking all possible permutations of the inequalities between the polynomials $p_{*k}(n_k)$, and substituting $\leq$ by the strict inequalities $<$ in some or all places, so that the resulting subsets of $\mathbb{N}^d$ do not have common points, and partition $\mathbb{N}^d$.

Let us prove that for *a compactly supported* sequence f with $\|f\|_{\mathcal{S}^d} \leq 1$ we have

$$|S_{\omega_m}| \overset{r,d}{\ll} \min\left(1, p_{*d}^{-\rho}(m)\right), \ m \in \mathbb{N}, \quad S_{\omega_m} := \sum_{\mathbf{n} \in \omega_m} f_{\mathbf{n}} B_{\mathbf{n}}(\vec{p}). \tag{12}$$

For brevity let $n_1 := n, \ p_1 := p$, and for $\mathbf{n} = (n, n_2, \dots, n_d) \in \mathbb{N}^d, \ m \in \mathbb{N}, \ k = 1, \dots, d-1$, denote $\mathbf{n}^k := (n_{k+1}, \dots, n_d) \in \mathbb{N}^{d-k}$,

$$B_{\mathbf{n}^k}(\vec{p}) := \prod_{l=k+1}^{d} B_{n_l}(p_l), \quad \Pi_{\mathbf{n}^k}(\vec{p}) := \prod_{l=k+1}^{d} \frac{\min(1, p_{*l}(n_l))}{n_l};$$
$$\omega_m^k := \left\{\mathbf{n}^k \in \mathbb{N}^{d-k} : \ p_{*k+1}(n_{k+1}) \geq \dots \geq p_{*d}(n_d), \ n_d \geq m\right\}. \tag{13}$$

Then

$$S_{\omega_m} = \sum_{\mathbf{n}^1 \in \omega_m^1} B_{\mathbf{n}^1}(\vec{p}) \left(\sum_{n:\, p_*(n) \geq p_{*2}(n_2)} f_{\mathbf{n}} B_n(p) \right), \tag{14}$$

and according to (10), (8),

$$\left| \sum_{n:\, p_*(n) \geq p_{*2}(n_2)} f_{\mathbf{n}} B_n(p) \right| \overset{r}{\ll} \min\left(1, p_{*2}^{-\rho}(n_2)\right), \quad |B_{\mathbf{n}^1}(\vec{p})| \overset{d}{\ll} \Pi_{\mathbf{n}^1}(\vec{p}).$$

Therefore, (12) is a corollary of the following chain of inequalities:

$$|S_{\omega_m^d}| \overset{r,d}{\ll} \sum_{\mathbf{n}^1 \in \omega_m^1} \Pi_{\mathbf{n}^1}(\vec{p}) \min\left(1, p_{*2}^{-\rho}(n_2)\right)$$
$$= \sum_{\mathbf{n}^2 \in \omega_m^2} \Pi_{\mathbf{n}^2}(\vec{p}) \left(\sum_{n_2:\, p_{*2}(n_2) \geq p_{*3}(n_3)} \frac{\min\left(p_{*2}(n_2), p_{*2}^{-\rho}(n_2)\right)}{n_2} \right)$$

$$\overset{r}{\ll} \sum_{\mathbf{n}^2 \in \omega_m^2} \Pi_{\mathbf{n}^2}(\vec{p}) \min\left(1, {p_*}_3^{-\rho}(n_3)\right) \quad \cdots$$

$$\overset{r,d}{\ll} \sum_{n_d=m}^{\infty} \frac{\min\left({p_*}_d(n_d), {p_*}_d^{-\rho}(n_d)\right)}{n_d} \overset{r}{\ll} \min\left(1, {p_*}_d^{-\rho}(m)\right). \tag{15}$$

Clearly, (15) implies the global boundedness of the sums $S(f,\vec{p})$ for arbitrary compactly supported coordinate-wise slow sequences f, $\|f\|_{\mathcal{S}^d} \le 1$:

$$\sup_{\vec{p} \in \mathcal{P}^{r,d}} |S(f,\vec{p})| \overset{r,d}{\ll} 1.$$

(15) also implies the convergence of the infinite series in the sense of Pringsheim, if f is not compactly supported. Moreover, the following estimate of the rate of convergence of the sequence of the rectangular partial sums is a corollary from (15):

$$S_{\mathbf{m}}(f,\vec{p}) := \sum_{\mathbf{n} \in \Box_{\mathbf{m}}} f_{\mathbf{n}} B_{\mathbf{n}}(\vec{p}), \quad \mu = \mu(\mathbf{m}) := \min_k m_k;$$

$$|S(f,\vec{p}) - S_{\mathbf{m}}(f,\vec{p})| \overset{r,d}{\ll} \min\left(1, \max_{1 \le k \le d} {p_*}_k^{-\rho}(\mu)\right). \tag{16}$$

<u>Case 3</u>. Here $\mathcal{A} = \{1\}$, and we need to consider the purely discrete $1d$ series of exponential sums

$$S(f,p) = S(f,\{1\},p) := \sum_{n \in \mathbb{N}} f_n A_n(p).$$

As above, applying the Abel's transformation, for the m-th partial sum of this series we have

$$S_m(f,p) := \sum_{n=1}^{m} f_n A_n(p) = \sum_{n=1}^{m} f_n (T_n(p) - T_{n-1}(p)) = f_m T_m(p) + \sum_{n=1}^{m-1} T_n(p) \Delta f_n.$$

Therefore, by virtue of theorem 2 and (4,a), in this case we need to prove that if f is a slow sequence, then

$$\sup_{p \in \mathcal{P}^r} \sum_{n=1}^{\infty} |T_n(p) \Delta f_n| < \infty. \tag{17}$$

Although the proof of this result is essentially the same as that of (4,a) in the paper [1], we will reproduce here the most essential details. We do so because these details are needed for the complete proof of theorem 4, and also for the sake of the reader's convenience.

Let us first provide an outline of the most important features of the construction. Given a polynomial $p \in \mathcal{P}^r$, the domain of summation $\mathbb{N}$ is partitioned into 2 disjoint sets, both disjoint unions of intervals of natural numbers:

$$\begin{aligned} &\mathbb{N} = \mathbb{N}_1(p) \cup \mathbb{N}_2(p), \\ &\mathbb{N}_1(p) = \bigcup_{j\geq 1}[\mu_j, \nu_j], \quad \mathbb{N}_2(p) = \bigcup_{j\geq 1}(\nu_j, \mu_{j+1}), \quad \mu_j = \mu_j(p),\ \nu_j = \nu_j(p). \end{aligned} \tag{18}$$

This partitioning is accomplished according to *arithmetical* properties of the coefficient vector $\mathbf{x}$ of the polynomial $\frac{p}{2\pi} = P(\mathbf{x}, \cdot)$. Namely, it depends upon the approximation of $\mathbf{x}$ by vectors with the rational coordinates.

For $n \in [\mu_j, \nu_j]$, the point $\mathbf{x}$ is close to a rational point $\mathbf{y}_j$ with the "relatively small" denominator. Here, the exponential sums $T_n,\ A_n$ admit the *asymptotic formulas* of the type:

$$\begin{aligned} &T_n(p) = \sigma_j I_n(\tilde{p}_j) + \varepsilon_n,\ A_n(p) = \sigma_j B_n(\tilde{p}_j) + (\varepsilon_n - \varepsilon_{n-1}), \quad n \in [\mu_j, \nu_j]; \\ &\sigma_j = \sigma_j(\mathbf{x}), \quad \sum_j |\sigma_j| \overset{r}{\ll} 1; \quad \tilde{p}_j \in \mathcal{P}^r; \quad \delta_n \overset{r}{\ll} n^{-\alpha},\ \alpha = \alpha(r) > 0. \end{aligned} \tag{19}$$

On the contrary, if $n \in \mathbb{N}_2(p)$, the point $\mathbf{x}$ keeps away from the rational vectors with relatively small denominators. For such n, the following estimates are true

$$T_n(p) = \varepsilon_n,\ A_n(p) = \varepsilon_n - \varepsilon_{n-1}, \quad |\varepsilon_n| \overset{r}{\ll} n^{-\beta},\ \beta = \beta(r) > 0. \tag{20}$$

Admitting that such a construction is possible, we estimate the sum in (17), making use of (19) and (20), as follows:

$$\begin{aligned} &\sum_{n\in[\mu_j,\nu_j]} |I_n(\tilde{p}_j)\Delta f_n| \leq \sup_{\tilde{p}\in\mathcal{P}^r} \sum_{n\in\mathbb{N}} |I_n(\tilde{p})\Delta f_n| \overset{r}{\ll} 1; \\ &\sum_n |T_n(p)\Delta f_n| = \sum_{n\in\mathbb{N}_1} |T_n(p)\Delta f_n| + \sum_{n\in\mathbb{N}_2} |T_n(p)\Delta f_n| \\ &\overset{r}{\ll} \sum_j |\sigma_j| \sum_{n\in[\mu_j,\nu_j]} |I_n(\tilde{p}_j)\Delta f_n| + \sum_n \left(n^{-\alpha} + n^{-\beta}\right)\Delta f_n \overset{r,\alpha,\beta}{\ll} \sum_j |\sigma_j| + 1 \overset{r,\alpha,\beta}{\ll} 1. \end{aligned} \tag{21}$$

To realize this construction, let us apply the circular method of Hardy – Littlewood – Vinogradov. This method provides asymptotic formulas and estimates for H. Weyl's exponential sums

$$E_n(p) = E_n(\mathbf{x}) := \frac{1}{n}\sum_{t=1}^{n} e^{ip(t)} = \frac{1}{n}\sum_{t=1}^{n} e^{2\pi i P(\mathbf{x},t)}.$$

If one of the co-ordinates $x_1, \ldots, x_r$ of the vector $\mathbf{x}$ is an irrational number, then according to the theorem of H. Weyl [17]

$$\lim_{n\to\infty} E_n(p) = 0.$$

For a rational point $\mathbf{y} \in \mathbb{Z}^r/\mathbb{N}^r$, let us denote $Q = Q(\mathbf{y})$ the least common multiple of the denominators of its coordinates in the reduced representation, and rewrite $\mathbf{y}$ in the form $\mathbf{y} = \frac{\mathbf{a}}{Q}$:

$$\mathbf{y} = \left(\frac{b_1}{q_1}, \cdots, \frac{b_r}{q_r}\right) \in \mathbb{Z}^r/\mathbb{N}^r; \quad \mathbf{b} \in \mathbb{Z}^r,\ \mathbf{q} \in \mathbb{N}^r,\ (b_s, q_s) = 1;$$

$$Q = Q(\mathbf{y}) := [q_1, \ldots, q_r], \quad \mathbf{y} = \left(\frac{a_1}{Q}, \cdots, \frac{a_r}{Q}\right) = \frac{\mathbf{a}}{Q},\ \mathbf{a} \in \mathbb{Z}^r,\ Q \in \mathbb{N}.$$

Given such a $\mathbf{y}$, let us denote $\sigma(\mathbf{y})$ the corresponding (normalized) complete rational exponential sum (Gauss' sum of higher order)

$$\sigma(\mathbf{y}) := \frac{1}{Q}\sum_{n=1}^{Q} e^{2\pi i P(\mathbf{y},n)} = \frac{1}{Q}\sum_{n=1}^{Q} e^{2\pi i \frac{a_1 n + \cdots + a_r n^r}{Q}}, \quad Q = Q(\mathbf{y}).$$

One has

$$\lim_{n\to\infty} E_n(p) = E_Q(p) = \sigma(\mathbf{y}), \quad p = 2\pi P(\mathbf{y}, \cdot),$$

and $\sigma(\mathbf{y})$ satisfy the estimate of Hua Loo-Keng (for the proof, see Chen Jing-run [3], or S.B. Stechkin [5])

$$|\sigma(\mathbf{y})| \overset{r}{\ll} Q^{-\rho}(\mathbf{y}), \quad \rho = \frac{1}{r}. \tag{22}$$

As in [1], we will apply the results of G.I. Arkhipov's [2] version of the circular method. Accordingly, the space $\mathbb{R}^r$ is subdivided for a fixed natural n into 2 sets, $\mathbb{R}^r = \mathcal{E}_n \cup \mathcal{F}_n$. The coefficient vector $\mathbf{x} \in \mathbb{R}^r$ of a polynomial $p = \frac{P(\mathbf{x},\cdot)}{2\pi} \in \mathcal{P}^r$ is allotted to $\mathcal{E}_n$ (or *major arc*), iff in a narrow rectangular neighborhood of $\mathbf{x}$ there is a rational point $\mathbf{y} = (y_1, \ldots, y_r)$ with a "relatively small" denominator $Q(\mathbf{y})$:

$$\max_{1\le s\le r} n^s |x_s - y_s| = \max_{1\le s\le r} n^s \left|x_s - \frac{a_s}{Q}\right| \le n^{0.3}; \quad Q = Q(\mathbf{y}) \le n^{0.3}. \tag{23}$$

The set of all $\mathbf{x} \in \mathbb{R}^r$ that do not possess this property, is by the definition $\mathcal{F}_n$ (*minor arc*).

The necessary elements of the construction (19) – (20) are contained in the following statements.

Lemma 1 (see G.I. Arkhipov [2], Lemma 7 and Lemma 6) 1) *If* $\mathbf{x} \in \mathcal{E}_n$ *and* $\mathbf{y}$ *is the rational point satisfying (23),* $\mathbf{z} := \mathbf{x} - \mathbf{y}$, *then*

$$E_n(p) = \sigma(\mathbf{y})\frac{1}{n}\int_0^n e^{i\tilde{p}(t)}\,dt + \varepsilon, \quad \tilde{p} := 2\pi P(\mathbf{z}, \cdot); \quad |\varepsilon| \le 9rQn^{-1} \overset{r}{\ll} n^{-0.7}. \tag{24}$$

2) *If* $\mathbf{x} \in \mathcal{F}_n$ *then*

$$|E_n(p)| \overset{r}{\ll} n^{-\beta}, \quad \beta \ge (8r^2(\ln r + 1.5\ln\ln r + 4.2))^{-1}. \tag{25}$$

Further, we obviously have

$$p(-t) = 2\pi \sum_{s=1}^{r} (-1)^s x_s t^s = 2\pi P(\mathbf{x}', t), \quad \mathbf{x}' := (-x_1, \ldots, (-1)^r x_r)\,,$$

and let us note the following relations that are crucial in the estimates of the discrete sums:

$$\sigma(\mathbf{y}) = \sigma(\mathbf{y}'), \quad \mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{Z}^r/\mathbb{N}^r,\ \mathbf{y}' := (-y_1, \ldots, (-1)^r y_r)\,.$$

Therefore, the next statement is a corollary of Lemma 1.

Lemma 2 1) *If* $\mathbf{x} \in \mathcal{E}_n$, $\mathbf{y}$ *– the rational point satisfying (23),* $\tilde{p} := 2\pi P(\mathbf{x} - \mathbf{y}, \cdot)$, *then*

$$T_n(p) = \sigma(\mathbf{y}) I_n(\tilde{p}) + \varepsilon_n, \quad |\varepsilon_n| \overset{r}{\ll} n^{-\alpha},\ \alpha = 0.7. \tag{26}$$

2) *If* $\mathbf{x} \in \mathcal{F}_n$ *then (see also (25))*

$$|T_n(p)| \overset{r}{\ll} n^{-\beta}. \tag{27}$$

Lemma 3 (See [1], p. 152). *If* $n > n_0 := 1024$ *then for each* $\mathbf{x} \in \mathcal{E}_n$ *the rational point* $\mathbf{y} = \frac{\mathbf{a}}{Q}$ *satisfying (23) is unique.*

Indeed, assume the contrary. Then there is a rational point $\mathbf{y}' = \frac{\mathbf{a}'}{Q'} \neq \frac{\mathbf{a}}{Q}$ such that

$$\max\left(Q',\ \max_{1\le s\le r} n^s \left| x_s - \frac{a'_s}{Q'} \right| \right) \le n^{0.3}\,.$$

Since $\mathbf{y} \neq \mathbf{y}'$, there exists $s \in [1, r]$ for which $\frac{a_s}{Q} \neq \frac{a'_s}{Q'}$, so that

$$\frac{1}{QQ'} \le \left| \frac{a_s}{Q} - \frac{a'_s}{Q'} \right| \le 2n^{0.3-s} \le 2n^{-0.7}, \quad \max(Q, Q') \le n^{0.3}.$$

If $n > 1024$, the latter estimates contradict each other, which proves the uniqueness of $\mathbf{y}$ for $n > n_0$.

Fix a polynomial $p = 2\pi P(\mathbf{x}, \cdot) \in \mathcal{P}^r$. As the natural number $n \ge n_0$ increases, $\mathbf{x}$ alternatively dwells either in $\mathcal{E}_n$, or in $\mathcal{F}_n$. Respectively, let us subdivide $\mathbb{N}$ into two subsets

$$\mathbb{N}_1(p) := \{n \ge n_0,\ \mathbf{x} \in \mathcal{E}_n\}, \quad \mathbb{N}_2(p) := [1, n_0) \cup \{n \ge n_0,\ \mathbf{x} \in \mathcal{F}_n\}. \tag{28}$$

Let us consider the collection $\mathcal{Y}(\mathbf{x}) := \{\mathbf{y}_1, \mathbf{y}_2, \ldots\}$ of pair-wise *distinct* positions in $\mathbb{R}^r$ that are successively occupied, as n increases, by the rational approximant $\mathbf{y} = \mathbf{y}^{(n)}(\mathbf{x})$ of $\mathbf{x}$, in accordance with (23). Thus, the set $\mathbb{N}_1(p)$ is a union of disjoint intervals of natural numbers $[\mu_j, \nu_j](p)$:

$$\mathbb{N}_1(p) = \bigcup_{j\ge1} [\mu_j, \nu_j](p), \quad [\mu_j, \nu_j](p) := \left\{ n \ge n_0,\ \mathbf{y}^{(n)}(\mathbf{x}) = \mathbf{y}_j = \frac{\mathbf{a}_j}{Q_j} \right\}. \tag{29}$$

Respectively, we have $\mathbb{N}_2(p) = \mathbb{N} \setminus \mathbb{N}_1(p)$.

Lemma 4 (see also [1], Lemma 3). *Every two consecutive intervals* $[\mu_j, \nu_j]$, $[\mu_{j+1}, \nu_{j+1}]$ *in (29) are "wide apart", and the denominators* Q_j *are "rapidly increasing":*

$$\mu_{j+1} \geq (0.5)^{\frac{10}{3}} \nu_j^{\frac{4}{3}}, \quad Q_{j+1} \geq 0.5 Q_j^{\frac{4}{3}} . \tag{30}$$

The proof is quite similar to that of Lemma 3. Since $\mathbf{y}_j \neq \mathbf{y}_{j+1}$, there is $s \in [1, r]$ such that

$$\frac{1}{Q_j Q_{j+1}} \leq \left| \frac{a_{s,j}}{Q_j} - \frac{a_{s,j+1}}{Q_{j+1}} \right| \leq \nu_j^{-0.7} + \nu_{j+1}^{-0.7} \leq 2\nu_j^{-0.7}.$$

Therefore, $(\mu_j \mu_{j+1})^{0.3} \geq Q_j Q_{j+1} \geq 0.5 \nu_j^{0.7}$, because $Q_j \leq \mu_j^{0.3}$, $Q_{j+1} \leq \mu_{j+1}^{0.3}$, and the estimates (30) easily follow.

We conclude that the asymptotic formulas (19) and the estimates (20) are true with $\sigma_j = \sigma(\mathbf{y}_j)$, and in view of (30), (22)

$$\sum_j |\sigma_j| \overset{r}{\ll} \sum_j Q_j^{-\rho} \overset{r}{\ll} 1,$$

which completes the consideration of the case 3.

Concerning the *rate of convergence* of the sequence of partial sums $S_m(f, p)$ in this case, the situation is quite naturally more complicated than in the previously considered cases, see (10) and (16). This rate depends on the arithmetical properties of the coefficient vector $\mathbf{x}$ of $P = p/2\pi$. For a positive number ε, in ε-neighborhood of $\mathbf{x}$ find the rational vector $\mathbf{y} \neq \mathbf{x}$ with the "smallest denominator" $Q(\mathbf{y})$, and denote the latter $Q(p, \varepsilon)$. Clearly, $Q(p, \varepsilon) \to \infty$, $\varepsilon \to 0$. The above construction implies that

$$|S(f, p) - S_m(f, p)| \overset{r}{\ll} Q^{-\rho}\left(p, m^{-0.3}\right) + m^{-\beta}.$$

For more details, the reader may be referred to [12].

<u>Case 4</u>. Let us again temporarily assume that the sequence $\{f_{\mathbf{n}}\}$ is compactly supported, and prove the estimate (5). We apply the induction in the number of elements $a := \sharp\mathcal{A}$ in $\mathcal{A}$, and make use of Telyakovskii's idea, i. e. partition the domain of summation into polynomial octants.

If $a = 0$, i. e. $\mathcal{A} = \emptyset$, the claim (5) is true, by the result of our consideration of the case 2. Notice that if we consider $A_n(p) = T_n(p) - T_{n-1}(p)$ as a function of the coefficients of the polynomial p for a fixed n, then this function is periodic in each of its variables, and the period $= 2\pi$. Therefore, we can assume, without loss of generality, that the coefficients of the polynomials p_k in $A_{n_k}(p_k)$ do not exceed π in the absolute value.

According to the principle of induction, let us assume that

$$\sup_{\vec{p} \in \mathcal{P}^{r,d}} |S(f, \mathcal{A}, \vec{p})| \overset{r,d}{\ll} 1, \quad \sharp\mathcal{A} = a, \tag{31}$$

and deduce from this assumption the bounds for the sums over the octants $\omega_m,\ m \in \mathbb{N}$, see (11):

$$S_{\omega_m}(f,\mathcal{A},\vec{p}) := \sum_{\mathbf{n}\in\omega_m} f_{\mathbf{n}} C_{\mathbf{n}}(\mathcal{A},\vec{p}), \quad \sharp\mathcal{A} = a+1.$$

Let us re-write these sums as in (14):

$$S_{\omega_m}(f,\mathcal{A},\vec{p}) = \sum_{\mathbf{n}^1\in\omega_m^{d-1}} C_{\mathbf{n}^1}(\vec{p}) D_{\mathbf{n}^1}(f,\mathcal{A},p), \tag{32}$$

where

$$C_{\mathbf{n}^1}(\mathcal{A},\vec{p}) := \prod_{k=2}^{d} C_{n_k}(\mathcal{A},p_k), \quad D_{\mathbf{n}^1}(f,\mathcal{A},p) := \sum_{n:\ p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}} C_n(\mathcal{A},p).$$

By virtue of (6), we can estimate the product $C_{\mathbf{n}^1}(\mathcal{A},\vec{p})$ trivially, see also (13):

$$|C_{\mathbf{n}^1}(\mathcal{A},\vec{p})| \overset{d}{\ll} \prod_{k=2}^{d} \frac{\min(1,p_{*k}(n_k))}{n_k} = \Pi_{\mathbf{n}^1}(\vec{p}). \tag{33}$$

As for the sums $D_{\mathbf{n}^1}(f,p)$, there are two possibilities.
(i) $1 \notin \mathcal{A}$, so that $C_n(\mathcal{A},p) = B_n(p)$. In this case, we can estimate $D_{\mathbf{n}^1}(f,p)$ and $S_{\omega_m}(f,\mathcal{A},\vec{p})$ using (33), (15), and avoiding a reference to the assumption of the induction (31):

$$|D_{\mathbf{n}^1}(f,\mathcal{A},p)| = \left|\sum_{n:\ p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}} B_n(p)\right| \overset{r}{\ll} \min\left(1, p_{*2}^{-\rho}(n_2)\right),$$

$$|S_{\omega_m}(f,\mathcal{A},\vec{p})| \overset{r,d}{\ll} \sum_{\mathbf{n}^1\in\omega_m^{d-1}} \Pi_{\mathbf{n}^1}(\vec{p}) \min\left(1, p_{*2}^{-\rho}(n_2)\right) \overset{r,d}{\ll} \min\left(1, p_{*d}^{-\rho}(m)\right). \tag{34}$$

(ii) $1 \in \mathcal{A}$, so that $C_n(\mathcal{A},p) = A_n(p)$, and

$$D_{\mathbf{n}^1}(f,\mathcal{A},p) = \sum_{n:\ p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}} A_n(p) = \sum_{n:\ p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}}(T_n(p) - T_{n-1}(p)). \tag{35}$$

In this case, we need to make use of the assumption, that all coefficients of the polynomial p are $\leq \pi$ in the absolute value. Accordingly, the condition $p_*(n) \geq p_{*2}(n_2)$ implies

$$n \gg p_{*2}^{\rho}(n_2), \quad n^{-\beta} \ll \min\left(1, p_{*2}^{-\gamma}(n_2)\right), \quad \gamma := \rho\beta. \tag{36}$$

Let us consider a numerical sequence $\{\varepsilon_n\}$ that satisfies the estimate $|\varepsilon_n| \ll n^{-\beta}$, and a sequence $f = \{f_n\} \in \mathcal{S},\ \|f\|_{\mathcal{S}} \leq 1$. Then, applying the Abel's transformation (see also (8)) we see that

$$\left|\sum_{n\geq m} f_n(\varepsilon_n - \varepsilon_{n-1})\right| = \left|\varepsilon_{m-1} f_m - \sum_{n\geq m} \varepsilon_n \Delta f_n\right| \overset{\beta}{\ll} m^{-\beta},$$

and consequently, it follows from (36) that

$$\left|\sum_{n:\,p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}}(\varepsilon_n-\varepsilon_{n-1})\right| \ll \min\left(1, p_{*2}^{-\gamma}(n_2)\right)$$

From here, arguing exactly as above, we obtain

$$\left|\sum_{\mathbf{n}^1\in\omega_m^{d-1}} C_{\mathbf{n}^1}(\mathcal{A},\vec{p})\left(\sum_{n:\,p_*(n)\geq p_{*2}(n_2)} f_{\mathbf{n}}(\varepsilon_n-\varepsilon_{n-1})\right)\right|$$
$$\overset{r,d}{\ll} \sum_{\mathbf{n}^1\in\omega_m^{d-1}} \Pi_{\mathbf{n}^1}(\vec{p})\min\left(1, p_{*2}^{-\gamma}(n_2)\right) \overset{r,d}{\ll} \min\left(1, p_{*d}^{-\gamma}(m)\right).$$

With the help of this estimate, we "clean up" the summation of the error terms ε_n of the asymptotic formulas (19), and also the summation over $n\in\mathbb{N}_2(p)$ in the sum (20). Up to a small error, the summation in n is localized to that of the main terms of the asymptotic formulas on the set $\mathbb{N}_1(p)=\bigcup_j[\mu_j,\nu_j](p)$. From (26), (27) we see that

$$\left|S_{\omega_m}(f,\mathcal{A},\vec{p})-\sum_j \sigma_j S\left(f^{(j,m)},\vec{p'}_j\right)\right| \overset{r,d}{\ll} \min\left(1, p_{*d}^{-\gamma}(m)\right), \tag{37}$$

where

$$\vec{p'}_j=(\tilde{p}_j,p_2,\ldots,p_d),\quad S\left(f^{(j,m)},\vec{p'}_j\right)=\sum_{\mathbf{n}\in\mathbb{N}^d} f_{\mathbf{n}}^{(j,m)}B_n(\tilde{p}_j)C_{\mathbf{n}^1}(\vec{p}),$$

$$f_{\mathbf{n}}^{(j,m)}:=\begin{cases} f_{\mathbf{n}} & \text{for } \mathbf{n}\in\omega_m,\ n\in[\mu_j,\nu_j](p),\\ 0 & \text{for all other } \mathbf{n}\in\mathbb{N}^d.\end{cases}$$

For each fixed $m,\ j\in\mathbb{N}$, the set $\{\mathbf{n}:\ \mathbf{n}\in\omega_m,\ n\in[\mu_j,\nu_j](p)\}$ is coordinate wise connected (possibly, empty), so that the sequence $f^{(j,m)}:=\left\{f_{\mathbf{n}}^{(j,m)}\right\}_{\mathbf{n}\in\mathbb{N}^d}$ is coordinate wise slow, and $\|f^{(j,m)}\|_{\mathcal{S}^d}\leq 2\|f\|_{\mathcal{S}^d}\leq 2$. Further, the number of A's in the product $B\prod_{k=2}^d C$ is by 1 less than that in the original product $\prod_{k=1}^d C$. Therefore, by the assumption of the induction (31)

$$\left|S\left(f^{(j,m)},\vec{p'}_j\right)\right| \overset{r,d}{\ll} 1. \tag{38}$$

Since we also have $\sum_j |\sigma_j| \overset{r}{\ll} 1$, it follows from (37) that

$$|S_{\omega_m}(f,\mathcal{A},\vec{p})| \overset{r,d}{\ll} 1,\quad \sharp\mathcal{A}=a+1.$$

Thereby, the global boundedness result of the whole sum for a compactly supported slow sequence f also follows:

$$\sup_{\vec{p}\in\mathcal{P}^{r,d}} |S(f,\mathcal{A},\vec{p})| < \infty.$$

Let us finish by proving the convergence of the series $S(f,\mathcal{A},\vec{p})$ for not compactly supported slow sequences f. To this end, it is sufficient to prove that for every fixed collection of polynomials $\vec{p}\in\mathcal{P}^{r,d}$

$$\sup_{f\in\mathcal{S}_0^d} |S_{\omega_m}(f,\mathcal{A},\vec{p})| \to 0, \quad m\to\infty,$$

where $\mathcal{S}_0^d$ denotes the set of all compactly supported slow sequences f with $\|f\|_{\mathcal{S}^d}\le 1$. Clearly, we can assume that none of the polynomials p_k is $\equiv 0$, so that the right sides of (34) (case (i)) and (36) (case (ii)) are uniformly small on $\mathcal{S}_0^d$. In particular, we can concentrate only on the case (ii).

Assume that one of the coordinates of the coefficient vector $\mathbf{x}$ of the polynomial $P(\mathbf{x},\cdot) = p(\cdot)/2\pi$ is an irrational number. Then for each fixed j, the set $\{\mathbf{n}\in\omega_m,\ n\in[\mu_j,\nu_j](p)\}$ is empty for all sufficiently large m, so that

$$f^{(j,m)} \equiv 0, \quad m\ge M(j).$$

Indeed, the condition $\mathbf{n}=(n,n_2,\dots,n_d)\in\omega_m$ implies, by the definition (11) of the octant ω_m, that $p_*(n)\ge p_{*d}(m)$. The latter contradicts the bound $n\le\nu_j(p)$, if m is large enough.

It follows that there exists a sequence $\{J(m)\}_{m\in\mathbb{N}}$ such that

$$J(m)\to\infty,\ m\to\infty;\quad S\left(f^{(j,m)},\vec{p'}_j\right)=0,\ j\le J(m),$$

so that by (37)

$$S_{\omega_m}(f,\mathcal{A},\vec{p}) \overset{r,d}{\ll} p_{*d}^{-\gamma}(m) + \left(\sum_{j>J(m)}\sigma_j\right) \to 0, \quad m\to\infty.$$

Finally, if $\mathbf{x}\in\mathbb{Z}^r/\mathbb{N}$, then the set $\mathbb{N}_1(p)=\bigcup_j[\mu_j,\nu_j]$ is concluded by the semi-axis $[\mu_J,\infty)$, and $\tilde{p}_J\equiv 0$, so that $n\ge\mu_J$ we simply have $B_n(\tilde{p}_J)\equiv 0$. Consequently, if m is sufficiently large, $S\left(f^{(j,m)},\vec{p'}_j\right)=0$ for all j, and for all such m we have

$$S_{\omega_m}(f,\mathcal{A},\vec{p}) \overset{r,d}{\ll} p_{*d}^{-\gamma}(m),$$

which concludes the proof.

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