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I will start this article from a short memorial section, very fragmentary and non-monumental. I hesitated a long time. It was by far not easy to decide whether to write this part or to abstain, and if to write – what form to select. Because our crowd is a poor belletrist, with very few exceptions to which I do not belong. We are trained to directly dive into definitions, notations, and theorems that we righteously think are absolutely important, of an almost eternal significance. We prefer to keep the inner eyes of our personal memories *comfortably shut*, and move on, and on, and prove that our ε is better than ... all other ε's.

Still, I will try. I will not touch Vasil's mathematics. Others described his works much better than I ever could. Instead I will attempt to write something on a human side, maybe funny, maybe, not. My memory has a warm corner dedicated to Vasil as a person and to things we experienced together. I will try to share just one non-sophisticated event with you.

Autumn of 1975, Warsaw. Zbigniew Cieselski did such a great job with the semester on approximation theory! I spent the whole term in the Banach Center. It was a full 3 months of really good time, with the freedom to choose either an excellent lecture on Mokotowska 25, or sitting at home over a proof of something important, or a stroll in the magnificent Lazienki Park, with Chopin's piano concerts in the open air. "The Godfather" was played in some movie theaters in Warsaw, and if one was patient enough to spend a few hours in a line early in the morning, before the store was open, – it was even possible to buy a record of "ABBA". Of course, we also partied a lot, and danced not only in mathematical circles.

So, once we with Vasil received an invitation to a birthday party of a young guy, let us *denote* him here as Pavel, a friend of one of my non-mathematical friends in Warsaw. On our way to his place, Pavel openly declared to us that he was a junior officer of intelligence, worked in a unit specializing in observation and analysis of political views. Naturally, this voluntary confession made Vasil and me a bit tense, especially Vasil... However, on our awkwardly silent way from the bus stop, Pavel pointed his finger first at a bushy grove, and asked: "Do you see that bush ?", and then immediately, not waiting for our reply, turned the finger to a house, still in a distance. "Do you see *that* window ?"

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"Yes, we do. So what ?" "A year ago, my ministry gave me a one-bedroom apartment in that house. We are heading there. And from time to time my boss borrows the key from me, to spend a few merry hours with this or other lady from our office. And I am a curious fellow, too. I have to be, after all. So, while the boss and his party are there, I am sitting in the bush with a binocular, trying to figure out – whom he brought this time. They are cautious, and they keep the curtains down in most cases. I have better luck if I am able to catch them still on their way there, but the boss is smart, too. He orders me some urgent paperwork in the office that keeps me busy after he leaves ! "

The ice was not completely broken yet, but the general atmosphere obviously improved after Pavel's story. The birthday party developed quite successfully and in a right direction for a few hours. It was a truly one-bedroom apartment, with a tiny table, four simple chairs, and the master's bed right next to the table. I do not think that the place even had a separate kitchen, but maybe, I am wrong. Anyhow, we celebrated Pavel's birthday, and everything was OK. After some time, we got engaged in energetic discussions of various subjects. They ranged from the quality of binoculars that should be used for observation of Pavel's boss' activities, to democracy and advantages or disadvantages of socialism over capitalism, and vice versa, in our countries (Bulgaria, Poland, USSR), and to approximation in Hausdorff metric and Riemann conjecture.

By the way, I remember a problem that Vasil again mentioned in our warm conversation at the birthday table. He said, it came from physicists, and if I remember correctly, he assigned it to academician Hristov. Assume that $f(x)$ is a periodic uni-modular complex-valued function, i. e. $f(x+1) \equiv f(x),\ |f(x)| \equiv 1$. Is such a function uniquely determined by the sequence of complex moduli $\{|\hat{f}(n)|\},\ n = 0, \pm 1, \dots$ of its Fourier coefficients $\hat{f}(n) := \int_0^1 f(x) e^{-2\pi i n x}\, dx$? Then I could not answer this question, and I do not know the answer now. Sure, Vasil picked on me for my ignorance in this problem. He claimed, that I would never be able to solve it. He was right, I suppose. Then he pressed for a somewhat different issue. He claimed, the exact value of a certain constant in a Hausdorff approximation problem is related with the Riemann conjecture, that he was going to attack and solve both problems in a near future. Naturally, I responded that, first of all, these problems are not related at all, and second, that he, Vasil, is capable of neither finding that constant nor solving the Riemann conjecture. Thus, we could not achieve a reasonable consensus by purely scientific methods. So I said: "I am better in arm wrestling than any other approximator !" "No, I am better!" - Vasil replied. "Certainly. Let's try !" We quickly cleared a corner on the table, and properly positioned the elbows. "One, two, three, go !" - Pavel commanded. "Crack !" – the leg of Vasil's chair could not withstand the fury of the attack, and we with Vasil collapsed onto the bed, "Crack, crack !" - the bed responded under our weight. It fell apart into tiny splinters ! Almost unbelievable, hard to expect it from such a presumably solid piece of furniture that served for security of the state...

The officer's chair and bed were completely destroyed, but somehow it helped to finish melting the remainders of ice between us and Pavel. The party had a happy ending. However, those mathematical problems remained open. Neither did we learn, who was better in arm wrestling – I or Vasil.

And now to mathematics. For a natural d, denote R^d the real d-dimensional Euclidean space of vectors $\mathbf{x} = (x_1, \dots, x_d)$. The inner product, the length, the unit ball, and the unit sphere in R^d are, respectively,

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + \cdots + x_d y_d, \quad |\mathbf{x}| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}, \quad \mathcal{B}^d := \{\mathbf{x} : \ |\mathbf{x}| \le 1\}, \quad \mathcal{S}^{d-1} := \{\mathbf{y} : \ |\mathbf{y}| = 1\}.$$

For a fixed $\mathbf{y} \in \mathcal{S}^{d-1}$, a *plane wave* propagating in the direction $\mathbf{y}$, is a function on R^d, of the type $W(\mathbf{x}) = w(\langle \mathbf{x}, \mathbf{y} \rangle)$, $\mathbf{x} \in R^d$, where $w = w(x)$, $x \in= R^1$ is a uni-variate function. w is called *profile* of the wave W. If w is a uni-variate algebraic polynomial, we call the corresponding W a *polynomial wave* on R^d.

Let us denote $\mathcal{P}^{d,n}$ the space of algebraic polynomials, of = degree n in d real variables, with complex coefficients

$$\mathcal{P}^{d,n} := \text{Span} \left\{ x_1^{k_1} \cdots x_d^{k_d} \right\}_{0 \le k_1 + \cdots + k_d \le n} .$$

For a polynomial

$$P(\mathbf{x}) = \sum_{0 \le k_1 + \cdots + k_d \le n} c_{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d} \in \mathcal{P}^{d,n}$$

let us denote $S(P, \mathbf{x})$ its senior part, i. e.

$$S(P, \mathbf{x}) = \sum_{k_1 + \cdots + k_d = n} c_{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d} .$$

The general problem of our concern is the following.

How many polynomial waves are needed to compose a given polynomial $P \in \mathcal{P}^{d,n}$, and how can one characterize the directions of propagation of such waves ?

More precisely, we are interested in representations with the smallest number of waves whose sum equals P:

$$N(P) := \min \left\{ N : \quad P(\mathbf{x}) \equiv \sum_{j=1}^{N} w_j(\langle \mathbf{x}, \mathbf{y}_j \rangle) \right\} . \tag{1}$$

In recent literature, plane waves are known as *ridge functions*; we call $N(P)$ *ridge number* of P. We will also say that a collection of vectors (not necessarily minimal) $Y = \{\mathbf{y}_j\}_1^N \subset \mathcal{S}^{d-1}$ is a *composition set for P*, iff P can be represented as a sum of polynomial waves propagating in the directions $\mathbf{y}_1, \dots, \mathbf{y}_N$.

Since composition sets in problem (1) are subjected to optimization for a given polynomial P, this problem is of *non-linear approximation* type. That algebraic polynomials play a special role as a tool of intermediate approximants in a general ridge approximation problem has been known long ago. Pioneering works in this direction belong to B. A. Vostrecov and M. A. Kreines, c. f. [1]–[3]. Fundamentality problems of ridge functions and their generalizations were considered later by A. Pinkus and V. Ya. Lin [5].

We confine our attention in this paper on bivariate polynomials, i. e. on the simplest case $d = 2$. In this case, the following results are known.

1) *Every* $(n+1)$*-element collection* $Y = \{\mathbf{y}_j\}_1^{n+1}$ *of pair-wise non co-linear directions is a composition set for all polynomials* $P \in \mathcal{P}^{2,n}$, cf. e. g. [4].

2) *For polynomials in* $P \in \mathcal{P}^{2,n}$ *with the real senior part* $S(P)$ *one has*

$$\max_{P \in \mathcal{P}^{2,n}, \Im S(P) \equiv 0} N(P) = n. \tag{2}$$

In the other words, for each $P \in \mathcal{P}^{2,n}$ with real senior part, there exists an n-element composition set $Y = \{\mathbf{y}_j\}_1^n$; on the other hand, there are real polynomials in $\mathcal{P}^{2,n}$ for which the number n cannot be reduced. This result was recently proved by A. Schinzel [6].

We provide an alternative proof, based on Chebyshev – Fourier analysis, of Schinzel's result. Simultaneously, we somewhat strengthen the latter result by retrieving an element of arbitrariness to composition sets.

Theorem 1 *Let* $Y = \{\mathbf{y}_j\}_{j=1}^n$ *be an arbitrary* n*-element collection of pair-wise non co-linear (unit) vectors, and* P *a bivariate polynomial of degree* n*, with the real senior part* $S(P)$*. Then one can rigidly turn* Y *by an angle* $\varphi = \varphi(P, Y) \in [0, \pi/n]$ *so that the rotated collection* Y^φ *is a composition set for* P*.*

Remark 1. This theorem is sharp: one cannot improve the estimate $N(P) \le n$ for polynomials with the real senior part, see also [6], or drop the condition $\Im S(P) = 0$.

A. *The complex polynomial* $P(\mathbf{x}) := (x_1 + ix_2)^n$, $i = \sqrt{-1}$, *is not a composition of* n *polynomial waves, so that for this polynomial* $N(P) = n + 1$.

B. *The real polynomials* $P(\mathbf{x}) := \Re(x_1 + ix_2)^n$, $Q(\mathbf{x}) := \Im(x_1 + ix_2)^n$ *are not compositions of* $n - 1$ *polynomial waves, so that for these polynomials* $N(P) = N(Q) = n$.

Proof. Let us start from a reduction of the problem to interpolation by translates of trigonometric Dirichlet kernels.

For this purpose, let us use the following integral representation (Chebyshev – Fourier expansion) for polynomials $P \in \mathcal{P}^{2,n}$ (c.f. [7], [4], [8], and [9] where Chebyshev – Fourier analysis is discussed also for higher dimension case $d \ge 3$):

$$P(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{m=0}^{n} (m+1) a_m(\vartheta) u_m(\langle \mathbf{x}, \mathbf{y}_\vartheta \rangle) \right) d\vartheta \, . \tag{3}$$

In (3), $\mathbf{y}_\vartheta := (\cos\vartheta, \sin\vartheta)$, $\vartheta \in R^1$; and for $m = 0, 1, \ldots$, u_m and $a_m(\vartheta) = a_m(P, \vartheta)$ denote, respectively, the m-th uni-variate Chebyshev polynomial of the second kind, and the m-th Chebyshev ridge momentum of $P \in \mathcal{P}^{2,n}$:

$$u_m(x) := \frac{\sin(m+1)\arccos x}{\sqrt{1-x^2}}, \quad x \in [-1, 1]; \quad a_m(\vartheta) := \frac{1}{\pi} \int_{\mathcal{B}^2} P(\mathbf{x}) u_m(\langle \mathbf{x}, \mathbf{y}_\vartheta \rangle) \, d\mathbf{x}.$$

The m-th Chebyshev momentum a_m is a trigonometric polynomial of degree m, and $a_m(\vartheta+\pi)\equiv(-1)^m a_m(\vartheta)$. Denote $\mathcal{T}_m^{\pm}$ the subspace of all trigonometric polynomials with this property, and $\mathcal{D}_m$ – the Dirichlet kernel of $\mathcal{T}_m^{\pm}$:

$$\mathcal{T}_m^{\pm} := \text{Span}\,\{e^{il\vartheta}\}_{|l|\le m(2)}, \quad \mathcal{D}_m(\vartheta) := \sum_{|l|\le m(2)} e^{il\vartheta} = \frac{\sin(m+1)\vartheta}{\sin\vartheta},$$

where $\{|l|\le m(2)\}$ is the set of integers l that satisfy $|l|\le m$ and $l\equiv m \pmod 2$. Obviously $\mathcal{D}_m\in\mathcal{T}_m^{\pm}$, and

$$a(\vartheta)\equiv\frac{1}{2\pi}\int_0^{2\pi} a(\varphi)\mathcal{D}_m(\vartheta-\varphi)\,d\varphi, \quad \forall a\in\mathcal{T}_m^{\pm}. \tag{4}$$

In particular, we have

$$u_m(\langle\mathbf{x},\mathbf{y}_\vartheta\rangle)\equiv\frac{1}{2\pi}\int_0^{2\pi} u_m(\langle\mathbf{x},\mathbf{y}_\varphi\rangle)\,\mathcal{D}_m(\vartheta-\varphi)\,d\varphi, \tag{5}$$

because for each fixed $\mathbf{x}$, $u_m(\langle\mathbf{x},\mathbf{y}_\vartheta\rangle)$, as a function of ϑ, is a trigonometric polynomial in $\mathcal{T}_m^{\pm}$.

Now let us consider a uni-variate function (in fact, an algebraic polynomial) $w(x)$, $x\in[-1,1]$, and expand it into Fourier series with regard to the system $\{u_m\}$:

$$w(x)=\sum_m b_m u_m(x), \quad b_m=b_m(w)=\frac{2}{\pi}\int_{-1}^{1} w(x)u_m(x)\sqrt{1-x^2}\,dx. \tag{6}$$

If $\mathbf{y}$ is a fixed unit vector, say, $\mathbf{y}=(\cos\vartheta_1,\sin\vartheta_1)=\mathbf{y}_{\vartheta_1}$, then it follows from (6) and (5) that the expansion (3) of a plane wave $W(\mathbf{x}):=w(\langle\mathbf{x},\mathbf{y}\rangle)$ in the direction y is given by

$$w(\langle\mathbf{x},\mathbf{y}_{\vartheta_1}\rangle)=\sum_m b_m u_m(\langle\mathbf{x},\mathbf{y}_{\vartheta_1}\rangle)=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_m (m+1)\frac{b_m\mathcal{D}_m(\vartheta_1-\vartheta)}{m+1}u_m(\langle\mathbf{x},\mathbf{y}_\vartheta\rangle)\right)d\vartheta\ .=$$

Therefore (c. f. also [8]), the momenta of a single plane wave are numerical multiples of shifted Dirichlet kernels:

$$a_m(w(\langle\cdot,\mathbf{y}_{\vartheta_1}\rangle),\vartheta)=\frac{b_m\mathcal{D}_m(\vartheta-\vartheta_1)}{m+1}, \quad m=0,1,\dots \tag{7}$$

Let us consider an N-element collection $\Theta=\{\vartheta_j\}_1^N$ of real numbers, pair-wise non congruent modπ, so that the vectors of the corresponding direction set $Y_\Theta=\{(\cos\vartheta_j,\sin\vartheta_j)\}_1^N$ are pair-wise non co-linear. Denote

$$\mathcal{R}(Y_\Theta):=\left\{\sum_1^N w_j\left(\left\langle\mathbf{x},\mathbf{y}_{\vartheta_j}\right\rangle\right)\right\}$$

the set of compositions of plane waves in the directions $\mathbf{y}_{\vartheta_j}$. It follows from (3) and (7) that $\mathcal{R}(Y_\Theta)$ is equivalently described in terms of Chebyshev momenta:

$$f \in \mathcal{R}(Y_\Theta) \Longleftrightarrow a_m(f) \in \mathcal{T}_m(\Theta), \quad = \mathcal{T}_m(\Theta) := \text{Span}\,\{\mathcal{D}_m\,(\cdot - \vartheta_j)\}_{j=1}^{=} N\,,\ m = 0, 1, \ldots, \tag{8}$$

i. e. the momenta of $\mathcal{R}(Y_\Theta)$ are linear combinations of shifted = Dirichlet kernels.

Since $\mathcal{D}_m\,(\cdot - \vartheta_j) \in \mathcal{T}_m^{\pm}$, we obviously have

$$\mathcal{T}_m(\Theta) \subset \mathcal{T}_m^{\pm}, \quad \dim \mathcal{T}_m(\Theta) \leq \min(\dim \mathcal{T}_m^{\pm}, \text{card}\,\Theta) = \min(m+1, N).$$

The following stronger statement concerning shift spaces of Dirichlet kernels is true:

$$a)\ \ \mathcal{T}_m(\Theta) = \mathcal{T}_m^{\pm} \ \text{ for } \ N \geq m+1; \quad b)\ \ \dim \mathcal{T}_m(\Theta) = \text{rank} = [\mathcal{D}_m\,(\vartheta_k - \vartheta_j)]_{j,k=1}^{N} = \min(m+1, N). \tag{9}$$

For the sake of completeness, let us outline the proof of these known properties, see e. g. [4], or [8].

Denote $\Pi = \Pi_{\Theta,m}$ the linear operator of orthogonal projection of $\mathcal{T}_m^{\pm}$ onto $\mathcal{T}_m(\Theta)$, in the sense of $L^2(0, 2\pi)$:

$$\Pi(a) = \arg \min_{b \in \mathcal{T}_m(\Theta)} \|a - b\|_{L^2(0,2\pi)}, \quad \|a\|_{L^2(0,2\pi)} := \sqrt{\frac{1}{2\pi} \int_0^{2\pi} |a(\vartheta)|^2 \, d\vartheta}\,.$$

For a given trigonometric polynomial $a \in \mathcal{T}_m^{\pm}$, its projection $\Pi(a)$ is characterized by the usual orthogonality relations in $L^2(0, 2\pi)$

$$\int_0^{2\pi} (a(\vartheta) - \Pi(a, \vartheta))\mathcal{D}_m(\vartheta - \vartheta_k)\, d\vartheta = 0, \quad k = 1, \ldots, N,$$

which by (4) means that the polynomials a and $P(a)$ coincide on Θ:

$$\Pi(a, \vartheta) = a(\vartheta), \quad \vartheta \in \Theta. \tag{10}$$

If the number of points in Θ is large, namely $N \geq m+1$, then (10) implies that $P(a) \equiv a$ for all $a \in \mathcal{T}_m^{\pm}$, which is the same as $\mathcal{T}_m^{\pm} = \mathcal{T}_m(\Theta)$. This easily follows, if we separately consider cases of even and odd m, and refer to the uniqueness of solution of trigonometric Lagrange interpolation problem.

Further, for a given polynomial $a \in \mathcal{T}_m^{\pm}$, its projection $\Pi(a)$ onto $\mathcal{T}_m^{\pm}$ is a linear combination of shifted Dirichlet kernels

$$\Pi(a) = \sum_{j=1}^{N} \alpha_j \mathcal{D}_m(\cdot - \vartheta_j),$$

and according to (10), the coefficients α satisfy the following system of N linear equations

$$\sum_{j=1}^{N} \alpha_j \mathcal{D}_m(\vartheta_k - \vartheta_j) = a(\vartheta_k), \quad k = 1, \ldots, N. \tag{11}$$

This system is consistent whenever the data on the right are point-values $a(\vartheta_k)$ of a trigonometric polynomial $a \in \mathcal{T}_m^{\pm}$. Consequently,

$$\dim \mathcal{T}_m(\Theta) = \operatorname{rank} \left[\mathcal{D}_m\left(\vartheta_k - \vartheta_j\right)\right]_{j,k=1}^{N} = \dim \left\{a(\vartheta_1), \ldots, a(\vartheta_N)\right\}_{a \in \mathcal{T}_m^{\pm}} ,$$

and the latter dimension equals $\min(N, m+1)$, which follows from Lagrange interpolation. This completes the proof of (9).

Consider a collection of n angles $\Theta = \{\vartheta_j\}_1^n$, pair-wise non congruent $\bmod \pi$, and assume that a is a trigonometric polynomial in $\mathcal{T}_n^{\pm}$ whose senior harmonic (highest term) is real, i.e.

$$a(\vartheta) = \rho \cos\left(n\vartheta + \psi\right) + b(\vartheta)$$

where ρ and ψ are some fixed real numbers, and $b \in \mathcal{T}_{n-2}^{\pm}$.

Let us prove that there exists a real number $= \varphi_0 = \varphi_0(\Theta, \psi)$ such that

$$a \in \mathcal{T}_n\left(\Theta^{\varphi_0}\right), \quad 0 \le \varphi_0 \le \frac{\pi}{n} \tag{12}$$

where Θ^{φ} denotes the rigid shift of Θ by φ, i. e. $\Theta^{\varphi} := \{\vartheta_j + \varphi\}_1^n$.

Remark 2. For the proof of (12), we will add an extra "interpolation" point θ_0 to the original n-point collection Θ, and later eliminate it by an appropriate translation, that depends on the phase of the senior harmonic of the polynomial a. This consideration is quite coherent with M. Riesz' [10] trigonometric interpolation formula with an even number of fundamental points. The latter is a known classical tool of the proof of S. Bernstein's inequality, see e. g. A. Zygmund [11], Ch. 10, Section 3.

According to (9,a) with $N = n,\ m = n-2$ we have $b \in \mathcal{T}_n\left(\Theta^{\varphi}\right)$ for *every* shift φ. Therefore, the lower degree polynomial b can be disregarded, and without loss of generality, we assume that $b \equiv 0$. Plainly, we can also assume that $\rho = 1$ and $\psi = 0$.

Let us add to Θ an extra point, say ϑ_0, non-congruent $\bmod \pi$ with any of $\vartheta_j,\ j = 1, \ldots, n$. Consider the enlarged $(n+1)$-element collection $\Theta_* := \{\vartheta_j\}_0^n$, and its rigid shifts Θ_*^{φ}. Then according to (9,a) we have $\cos n\vartheta \in \mathcal{T}_n(\Theta_*^{\varphi})$ for every shift φ, i. e.

$$\cos n\vartheta \equiv \sum_{j=0}^{n} \alpha_j \mathcal{D}_n\left(\vartheta - \varphi - \vartheta_j\right) \tag{13}$$

Here, the coefficients α_j are functions of φ, $\alpha_j = \alpha_j(\varphi)$; these coefficients, in accordance with (11) and (9,b) are uniquely defined by the system of $n+1$ linear equations

$$\cos n\vartheta_k^{\varphi} = \sum_{j=0}^{n} \alpha_j(\varphi) \mathcal{D}_n\left(\vartheta_k^{\varphi} - \vartheta_j^{\varphi}\right), \quad k = 0, 1, \ldots, n.$$

Since $\vartheta_k^\varphi = \vartheta_k + \varphi$, and $\vartheta_k^\varphi - \vartheta_j^\varphi = \vartheta_k - \vartheta_j$, this system is the same as

$$\cos n(\vartheta_k + \varphi) = \sum_{j=0}^{n} \alpha_j(\varphi)\mathcal{D}_n\left(\vartheta_k - \vartheta_j\right) .$$

The matrix of this system is non-singular and does not depend on φ. Hence the solutions $\alpha_j(\varphi)$ are functions of the kind $\alpha_j(\varphi) = \rho_j \cos\left(n\varphi + \psi_j\right)$, where $\rho_j,\ \psi_j$ are some real constants. Therefore, each $\alpha_j(\varphi)$ vanishes at a certain point on each interval of the length $\geq \pi/n$. In particular, the coefficient $\alpha_0(\varphi)$ in (13) vanishes at a certain point $\varphi_0 \in [0, \pi/n]$. This completes the proof of (12), and in view (8) and (9,a)– also the proof of the theorem.

Let conclude by the proof of the statements contained in Remark 1. Making use of (3) and (8) (c. f. also [8]) it is easy to see that it suffices to prove, respectively, that if a collection of points $\{\vartheta_j\}$ is non-degenerate, i. e. the numbers ϑ_j are pair-wise non-congruent $\pmod \pi$, then

$$a) \quad = e^{in\vartheta} \notin \text{Span}\left\{\mathcal{D}_n\left(\vartheta - \vartheta_j\right)\right\}_{j=1}^{n} ; \quad b) \quad \cos n\vartheta \notin \text{Span}\left\{\mathcal{D}_n\left(\vartheta - \vartheta_j\right)\right\}_{j=1}^{n-1} . \tag{14}$$

Assume, on the contrary, that

$$a) \quad e^{in\vartheta} \equiv \sum_{j=1}^{n} \alpha_j \mathcal{D}_n\left(\vartheta - \vartheta_j\right), \quad b) \quad \cos n\vartheta \equiv \sum_{j=1}^{n-1} \beta_j \mathcal{D}_n\left(\vartheta - \vartheta_j\right)$$

Since $\mathcal{D}_n(\vartheta) = e^{-in\vartheta} + e^{i(-n+2)\vartheta} + \cdots + e^{i(n-2)\vartheta} + e^{in\vartheta}$ the assumption a) would imply that for $z_j := e^{i\vartheta_j}$

$$1 = \sum_{j=1}^{n} \alpha_j z_j^{-n}, \quad 0 = \sum_{j=1}^{n} \alpha_j z_j^{l},\ l = -n+2, -n+4, \ldots, n-2, n,$$

which is inconsistent, because the $n \times n$ matrix $\{z_j^l\},\ j = 1, 2, \ldots, n,\ l = -n+2, -n+4, \ldots, n-2, n$ is non-singular. Indeed, if it were singular, then there would exist a non-trivial set of n numbers $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that the rational function

$$R(z) := \gamma_0 z^{-n+2} + \gamma_1 z^{-n+4} + \cdots + \gamma_{n-2} z^{n-2} + \gamma_{n-1} z^n = z^{-n+2} \sum_{k=0}^{n-1} \gamma_k z^{2k}$$

vanishes at the points $z_j,\ j = 1, 2, \ldots, n$. This assumption would imply that a non-trivial polynomial $Q(z) := \sum_{k=0}^{n-1} \gamma_k z^k$ of degree $n-1$ vanishes at n points $z = z_j^2 = e^{2i\vartheta_j},\ j = 1, 2, \ldots, n$. The latter is impossible because the numbers $2\vartheta_j$ are pairwise non-congruent mod2π, so that the points z_j^2 are pairwise distinct.

Finally, the assumption b) would imply that

$$\cos n\vartheta \equiv 2\sum_{j=1}^{n-1} \beta_j \cos n(\vartheta - \vartheta_j), \quad 0 \equiv \sum_{j=1}^{n-1} \beta_j \mathcal{D}_{n-2}(\vartheta - \vartheta_j)$$

which is inconsistent, because the $(n-1)$-element set of translates $\{\mathcal{D}_{n-2}(\vartheta - \vartheta_j)\}_{j=1}^{n-1}$ is linearly independent, see (9).

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