Greedy approximation with regard to bases and general minimal systems

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ABSTRACT. This paper is a survey which also contains some new results on the nonlinear approximation with regard to a basis or, more generally, with regard to a minimal system. Approximation takes place in a Banach or in a quasi-Banach space. The last decade was very successful in studying nonlinear approximation. This was motivated by numerous applications. Nonlinear approximation is important in applications because of its increased efficiency. Two types of nonlinear approximation are employed frequently in applications. Adaptive methods are used in PDE solvers. The m-term approximation considered here is used in image and signal processing as well as the design of neural networks. The basic idea behind nonlinear approximation is that the elements used in the approximation do not come from a fixed linear space but are allowed to depend on the function being approximated. The fundamental question of nonlinear approximation is how to construct good methods (algorithms) of nonlinear approximation. In this paper we discuss greedy type and thresholding type algorithms.

1. GREEDY ALGORITHMS WITH REGARD TO BASES

Let a Banach space X with a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$, be given. We consider the following theoretical greedy algorithm. For a given element $f \in X$ we consider the expansion

$$(1.1) \qquad f = \sum_{k=1}^{\infty} c_k(f, \Psi)\psi_k.$$

For an element $f \in X$ we call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \ldots$, of the positive integers decreasing and write $\rho \in D(f)$ if

$$(1.2) \qquad |c_{k_1}(f, \Psi)| \geq |c_{k_2}(f, \Psi)| \geq \ldots \quad .$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the m-th greedy approximant of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by formula

$$G_m(f) := G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f, \Psi)\psi_{k_j}.$$

We note that there is another natural greedy type algorithm based on ordering $\|c_k(f, \Psi)\psi_k\|$ instead of ordering absolute values of coefficients. Denote $\Lambda_m(f)$ a set of indices such that

$$\min_{k \in \Lambda_m(f)} \|c_k(f, \Psi)\psi_k\| \geq \max_{k \notin \Lambda_m(f)} \|c_k(f, \Psi)\psi_k\|.$$

We define $G_m^X(f, \Psi)$ by formula

$$G_m^X(f, \Psi) := S_{\Lambda_m(f)}(f, \Psi), \quad \text{where} \quad S_E(f) := S_E(f, \Psi) := \sum_{k \in E} c_k(f, \Psi)\psi_k.$$

It is clear that in the case of normalized basis ($\|\psi_k\| = 1$, $k = 1, 2, \ldots$) the above two greedy algorithms coincide.

In the case $X = L_p$ we will write p instead of L_p in notations. It is a simple algorithm which describes the theoretical scheme (it is not computationally ready) for m-term approximation of an element f. We will call this algorithm Thresholding Greedy Algorithm (TGA). In order to understand the efficiency of this algorithm we compare its accuracy with the best possible when an approximant is a linear combination of m terms from Ψ. We define the best m-term approximation with regard to Ψ as follows

$$\sigma_m(f) := \sigma_m(f, \Psi)_X := \inf_{c_k, \Lambda} \|f - \sum_{k \in \Lambda} c_k \psi_k\|_X,$$

where inf is taken over coefficients c_k and sets of indices Λ with cardinality $|\Lambda| = m$. The best we can achieve with the algorithm G_m is

$$\|f - G_m(f, \Psi, \rho)\|_X = \sigma_m(f, \Psi)_X,$$

or a little weaker

(1.3)
$$\|f - G_m(f, \Psi, \rho)\|_X \leq G\sigma_m(f, \Psi)_X$$

for all elements $f \in X$ with a constant $G = C(X, \Psi)$ independent of f and m. It is clear that in the case $X = H$ is a Hilbert space and Ψ is an orthonormal basis we have

$$\|f - G_m(f, \Psi, \rho)\|_H = \sigma_m(f, \Psi)_H.$$

Let us begin our discussion by an important class of bases: wavelet type bases. Denote $\mathcal{H} := \{H_k\}_{k=1}^{\infty}$ the Haar basis on $[0, 1)$ normalized in $L_2(0, 1)$. We denote by $\mathcal{H}_p := \{H_{k,p}\}_{k=1}^{\infty}$ the Haar basis $\mathcal{H}$ renormalized in $L_p(0, 1)$. We will use the following definition of the L_p-equivalence of bases. We say that $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is L_p-equivalent to $\Phi = \{\phi_k\}_{k=1}^{\infty}$ if for any finite set Λ and any coefficients c_k, $k \in \Lambda$, we have

$$C_1(p, \Psi, \Phi)\|\sum_{k \in \Lambda} c_k \phi_k\|_p \leq \|\sum_{k \in \Lambda} c_k \psi_k\|_p \leq C_2(p, \Psi, \Phi)\|\sum_{k \in \Lambda} c_k \phi_k\|_p$$

with two positive constants $C_1(p, \Psi, \Phi), C_2(p, \Psi, \Phi)$ which may depend on p, Ψ, and Φ. For sufficient conditions on Ψ to be L_p-equivalent to $\mathcal{H}$ see [FJ] and [DKT]. In particular, it is known that all reasonable univariate wavelet type bases are L_p-equivalent to $\mathcal{H}$ for $1 < p < \infty$. We proved the following theorem in [T2].

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Theorem 1.1. *Let $1 < p < \infty$ and a basis Ψ be L_p-equivalent to the Haar basis $\mathcal{H}$. Then for any $f \in L_p(0,1)$ we have*

$$\|f - G_m^p(f, \Psi)\|_p \leq C(p, \Psi)\sigma_m(f, \Psi)_p$$

with a constant $C(p, \Psi)$ independent of f and m.

By a simple renormalization argument one obtains the following version of Theorem 1.1.

Theorem 1.1A. *Let $1 < p < \infty$ and a basis Ψ be L_p-equivalent to the Haar basis $\mathcal{H}_p$. Then for any $f \in L_p(0,1)$ and any $\rho \in D(f)$ we have*

$$\|f - G_m(f, \Psi, \rho)\|_p \leq C(p, \Psi)\sigma_m(f, \Psi)_p$$

with a constant $C(p, \Psi)$ independent of f, ρ, and m.

We note that [T2] also contains a generalization of Theorem 1.1 to the multivariate Haar basis obtained by the multiresolution analysis procedure. These theorems motivated us to consider the general setting of greedy approximation in Banach spaces. We concentrated on studying bases which satisfy (1.3) for all individual functions.

Definition 1.1. *We call a basis Ψ greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$(1.4) \qquad\qquad \|f - G_m(f, \Psi, \rho)\|_X \leq G\sigma_m(f, \Psi)_X$$

holds with a constant independent of f, m.

The following proposition has been proved in [KT1].

Proposition 1.1. *If Ψ is a greedy basis then (1.4) holds for any permutation $\rho \in D(f)$.*

Theorem 1.1A shows that each basis Ψ which is L_p-equivalent to the univariate Haar basis $\mathcal{H}_p$ is a greedy basis for $L_p(0,1)$, $1 < p < \infty$. We note that in the case of Hilbert space each orthonormal basis is a greedy basis with a constant $G = 1$ (see (1.4)).

We give now the definitions of unconditional and democratic bases.

Definition 1.2. *A basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ of a Banach space X is said to be unconditional if for every choice of signs $\theta = \{\theta_k\}_{k=1}^{\infty}$, $\theta_k = 1$ or -1, $k = 1, 2, \ldots$, the linear operator M_θ defined by $M_\theta(\sum_{k=1}^{\infty} a_k\psi_k) = \sum_{k=1}^{\infty} a_k\theta_k\psi_k$ is a bounded operator from X into X.*

Definition 1.3. *We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a democratic basis for X if there exists a constant $D := D(X, \Psi)$ such that for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$ we have $\|\sum_{k \in P} \psi_k\| \leq D\|\sum_{k \in Q} \psi_k\|$.*

We proved in [KT1] the following theorem.

Theorem 1.2. *A basis is greedy if and only if it is unconditional and democratic.*

This theorem gives a characterization of greedy bases. Further investigations ([T3], [CDH], [KP], [GN]) showed that the concept of greedy bases is very useful in direct and inverse theorems of nonlinear approximation and also in applications in statistics. The papers [KT1], [T2], contain other results on greedy bases.

Let us discuss a question of weakening the property of a basis of being a greedy basis. We begin with a concept of quasi-greedy basis.

Definition 1.4. *We call a basis Ψ quasi-greedy basis if for every $f \in X$ and every permutation $\rho \in D(f)$ we have*

$$(1.5) \qquad\qquad \|G_m(f, \Psi, \rho)\|_X \leq C\|f\|_X$$

with a constant C independent of f, m, and ρ.

It is clear that (1.5) is weaker then (1.4). P. Wojtaszczyk [W] proved the following theorem.

Theorem 1.3. *A basis Ψ is quasi-greedy if and only if for any $f \in X$ and any $\rho \in D(f)$ we have*

$$(1.6) \qquad\qquad \|f - G_m(f, \Psi, \rho)\| \to 0 \quad as \quad m \to \infty.$$

We proceed to an intermediate concept of almost greedy basis. This concept has been introduced and studied in [DKKT]. Let

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We define the following expansional best m-term approximation of f

$$\tilde{\sigma}_m(f) := \tilde{\sigma}_m(f, \Psi) := \inf_{\Lambda, |\Lambda|=m} \|f - \sum_{k \in \Lambda} c_k(f)\psi_k\|.$$

It is clear that

$$\sigma_m(f, \Psi) \leq \tilde{\sigma}_m(f, \Psi).$$

It is also clear that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f, \Psi) \leq C\sigma_m(f, \Psi).$$

Definition 1.5. *We call a basis Ψ almost greedy basis if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that*

$$(1.7) \qquad\qquad \|f - G_m(f, \Psi, \rho)\|_X \leq C\tilde{\sigma}_m(f, \Psi)_X$$

holds with a constant independent of f, m.

The following proposition follows from the proof of Theorem 3.3 of [DKKT] (see Theorem 1.4 below).

Proposition 1.2. *If Ψ is an almost greedy basis then (1.7) holds for any permutation $\rho \in D(f)$.*

The following characterization of almost greedy bases has been obtained in [DKKT].

Theorem 1.4. *Suppose Ψ is a basis of a Banach space. The following are equivalent:*
   A. Ψ is almost greedy.
   B. Ψ is quasi-greedy and democratic.
   C. For any (respectively, every) $\lambda > 1$ there is a constant $C = C_\lambda$ such that

$$\|f - G_{[\lambda m]}(f, \Psi)\| \leq C_\lambda \sigma_m(f, \Psi).$$

We will prove an estimate for $\tilde{\sigma}_n(f, \Psi)$ in terms of $\sigma_m(f, \Psi)$ for a quasi-greedy basis Ψ. For a basis Ψ we define the fundamental function

$$\varphi(m) := \sup_{|A| \leq m} \|\sum_{k \in A} \psi_k\|.$$

We also need the following function

$$\phi(m) := \inf_{|A| = m} \|\sum_{k \in A} \psi_k\|.$$

It will be convenient to define the quasi-greedy constant K to be the least constant such that

$$\|G_m(f)\| \leq K\|f\| \quad \text{and} \quad \|f - G_m(f)\| \leq K\|f\|, \quad f \in X.$$

We will prove an inequality that has been obtained in [DKKT].

Theorem 1.5. *Let Ψ be a quasi-greedy basis. Then for any m and r there exists a set E, $|E| \leq m + r$ such that*

$$\|f - S_E(f)\| \leq C\big(1 + \frac{\varphi(m)}{\phi(r+1)}\big)\sigma_m(f).$$

Proof. If $\sigma_m(f) = 0$ then $f = \sum_{k \in A} c_k(f)\psi_k$, $|A| \leq m$ and, therefore, $S_A(f) = f$. Let $\sigma_m(f) \neq 0$ and A be a set, $|A| = m$, such that

(1.8) $$\|f - p_m(f)\| \leq 2\sigma_m(f), \quad p_m(f) = \sum_{k \in A} b_k \psi_k.$$

Denote $g := f - p_m(f)$. Let B, $|B| = r$, be such that

$$G_r(g) = \sum_{k \in B} c_k(g)\psi_k.$$

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Consider

(1.9) $$f - S_{A \cup B}(f) = g - S_{A \cup B}(g) = g - S_B(g) - S_{A \setminus B}(g).$$

By the assumption that Ψ is quasi-greedy and by the definition of B we get

(1.10) $$\|g - S_B(g)\| \le C_1 \|g\| \le 2C_1 \sigma_m(f).$$

Let us estimate $\|S_{A \setminus B}(g)\|$. By Lemma 2.2 from [DKKT] we get

$$\max_{k \in A \setminus B} |c_k(g)| \le 4K^2 (\phi(r+1))^{-1} \|g\|.$$

Next, by Lemma 2.1 from [DKKT] we obtain

(1.11) $$\|S_{A \setminus B}(g)\| \le (2K)^3 \varphi(m) \phi(r+1)^{-1} \|g\|.$$

Combining (1.10) and (1.11) we derive from (1.9) for $E := A \cup B$

$$\|f - S_E(f)\| \le C \Big(1 + \frac{\varphi(m)}{\phi(r+1)}\Big) \sigma_m(f).$$

Theorem 1.5 is proved.

2. Weak Greedy Algorithms with regard to bases

The following weak type greedy algorithm was considered in [T2]. Let $t \in (0,1]$ be a fixed parameter. For a given basis Ψ and a given $f \in X$ denote $\Lambda_m(t)$ any set of m indices such that

(2.1) $$\min_{k \in \Lambda_m(t)} |c_k(f, \Psi)| \ge t \max_{k \notin \Lambda_m(t)} |c_k(f, \Psi)|$$

and define

$$G_m^t(f) := G_m^t(f, \Psi) := \sum_{k \in \Lambda_m(t)} c_k(f, \Psi) \psi_k.$$

We call it the Weak Thresholding Greedy Algorithm (WTGA) with the weakness sequence $\{t\}$. It was proved in [T2] that in the case of $X = L_p$, $1 < p < \infty$, and Ψ is the Haar system $\mathcal{H}_p$ normalized in L_p we have for any $f \in L_p$

(2.2) $$\|f - G_m^t(f, \mathcal{H}_p)\|_{L_p} \le C(p,t) \sigma_m(f, \mathcal{H}_p)_{L_p}.$$

We note here that the proof of (2.1) from [T2] works for any greedy basis instead of the Haar system $\mathcal{H}_p$. Thus for any greedy basis Ψ of a Banach space X and any $t \in (0,1]$ we have for each $f \in X$

(2.3) $$\|f - G_m^t(f, \Psi)\|_X \le C(\Psi, t) \sigma_m(f, \Psi)_X.$$

This means that for greedy bases we have more flexibility in constructing near best m-term approximants.

We now consider the Weak Thresholding Greedy Algorithm with regard to a quasi-greedy basis Ψ. The following theorem is essentially due to Wojtaszczyk [W].

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Theorem 2.1. *Let Ψ be a quasi-greedy basis for a Banach space X. Then for any fixed $t \in (0,1]$ we have for each $f \in X$ that*

$$G_m^t(f, \Psi) \to f \quad as \quad m \to \infty.$$

Proof. Let
$$G_m^t(f, \Psi) = \sum_{j \in \Lambda_m(t)} c_j(f) \psi_j = S_{\Lambda_m(t)}(f, \Psi).$$

Denote
$$\alpha := \max_{j \notin \Lambda_m(t)} |c_j(f)|$$

and
$$\Lambda_m^1 := \{j : |c_j(f)| > \alpha\} \subseteq \Lambda_m(t),$$

$$\Lambda_m^2 := \{j : |c_j(f)| \geq t\alpha\} \supseteq \Lambda_m(t).$$

Thus we have
$$S_{\Lambda_m(t)}(f, \Psi) = S_{\Lambda_m^1}(f, \Psi) + S_{\Lambda_m(t) \setminus \Lambda_m^1}(f, \Psi).$$

The assumption that Ψ is quasi-greedy implies that

$$S_{\Lambda_m^1}(f, \Psi) \to f \quad as \quad m \to \infty.$$

We will prove that
$$\|S_{\Lambda_m(t) \setminus \Lambda_m^1}(f, \Psi)\| \to 0 \quad as \quad m \to \infty.$$

We note that

(2.4) $$S_{\Lambda_m(t) \setminus \Lambda_m^1}(f, \Psi) = S_{\Lambda_m(t) \setminus \Lambda_m^1}\Big(\sum_{j : t\alpha \leq |c_j(f)| \leq \alpha} c_j(f) \psi_j, \Psi \Big).$$

We need a lemma on properties of quasi-greedy systems.

Lemma 2.1. *Let Ψ be a quasi-greedy basis. Then for any two finite sets of indices $A \subseteq B$ and coefficients $0 < t \leq |a_j| \leq 1$, $j \in B$, we have*

$$\|\sum_{j \in A} a_j \psi_j\| \leq C(X, \Psi, t) \|\sum_{j \in B} a_j \psi_j\|.$$

Proof. The proof is based on the following known lemma (see [DKKT]) that is essentially due to Wojtaszczyk [W].

Lemma 2.2. *Suppose Ψ is a quasi-greedy basis with a quasi-greedy constant K. Then for any real numbers a_j and any finite set of indecies P we have*

$$(4K^2)^{-1} \min_{j \in P} |a_j| \| \sum_{j \in P} \psi_j \| \leq \| \sum_{j \in P} a_j \psi_j \| \leq 2K \max_{j \in P} |a_j| \| \sum_{j \in P} \psi_j \|.$$

Using this lemma, we get

$$\| \sum_{j \in A} a_j \psi_j \| \leq 2K \| \sum_{j \in A} \psi_j \| \leq (2K)^2 \| \sum_{j \in B} \psi_j \| \leq (2K)^4 t^{-1} \| \sum_{j \in B} a_j \psi_j \|.$$

This proves Lemma 2.1.

We continue the proof of Theorem 2.1. Denote

$$f_\alpha := \sum_{j : t\alpha \leq |c_j(f)| \leq \alpha} c_j(f) \psi_j.$$

Then by Lemma 2.1 we get from (2.4)

$$\| S_{\Lambda_m(t) \setminus \Lambda_m^1}(f, \Psi) \| \leq C \| f_\alpha \|.$$

It remains to remark that $\alpha \to 0$ as $m \to \infty$ and $f_\alpha \to 0$ as $\alpha \to 0$.

We note that the mth greedy approximant $G_m(f, \Psi)$ changes if we renormalize the system $\{\psi_n\}$ (replace it by a system $\{\lambda_n \psi_n\}$). This gives us more flexibility in adjusting a given system $\{\psi_n\}$ for greedy approximation.

Let us now proceed to an almost greedy basis Ψ. Similarly to the proof of Theorem 2.1 one can prove the following lemma.

Lemma 2.3. *Let Ψ be a quasi-greedy basis. Then for a fixed $t \in (0, 1]$ and any m we have for any $f \in X$*
$$\| G_m^t(f, \Psi) \| \leq C(t) \| f \|.$$

Theorem 2.2. *Let Ψ be an almost greedy basis. Then for $t \in (0, 1]$ we have for any m*

$$(2.5) \qquad \| f - G_m^t(f) \| \leq C(t) \tilde\sigma_m(f).$$

Proof. Take any $\epsilon > 0$ and find P, $|P| = m$ such that

$$\| f - S_P(f) \| \leq \tilde\sigma_m(f) + \epsilon.$$

Let $Q := \Lambda_m(t)$ with $\Lambda_m(t)$ from the definition of $G_m^t(f)$. Then

$$(2.6) \qquad \| f - G_m^t(f) \| \leq \| f - S_P(f) \| + \| S_P(f) - S_Q(f) \|.$$

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We have

$$(2.7) \qquad S_P(f) - S_Q(f) = S_{P \setminus Q}(f) - S_{Q \setminus P}(f).$$

Let us estimate first $\|S_{Q \setminus P}(f)\|$. Denote $f_1 := f - S_P(f)$. Then

$$S_{Q \setminus P}(f) = S_{Q \setminus P}(f_1).$$

Next

$$\min_{k \in Q \setminus P} |c_k(f_1)| = \min_{k \in Q \setminus P} |c_k(f)| \geq \min_{k \in Q} |c_k(f)| \geq$$

$$t \max_{k \notin Q} |c_k(f)| \geq t \max_{k \notin Q} |c_k(f_1)| = t \max_{k \notin Q \setminus P} |c_k(f_1)|.$$

Thus $Q \setminus P = \Lambda_n(t)$ for f_1 with $n := |Q \setminus P|$. By Lemma 2.3 we have

$$(2.8) \qquad \|S_{Q \setminus P}(f)\| \leq C_1(t)\|f_1\|.$$

We now estimate $\|S_{P \setminus Q}(f)\|$. From the definition of Q we easily derive

$$(2.9) \qquad at \leq b \quad \text{where} \quad a := \max_{k \in P \setminus Q} |c_k(f)|, \quad b := \min_{k \in Q \setminus P} |c_k(f)|.$$

By Lemma 2.2 (see Lemma 2.1 from [DKKT])

$$(2.10) \qquad \|S_{P \setminus Q}(f)\| \leq 2Ka\| \sum_{k \in P \setminus Q} \psi_k\|$$

and (see Lemma 2.2 from [DKKT])

$$(2.11) \qquad \|S_{Q \setminus P}(f)\| \geq (4K^2)^{-1}b\| \sum_{k \in Q \setminus P} \psi_k\|.$$

By Theorem 1.4 an almost greedy basis is a democratic basis. Thus we get

$$(2.12) \qquad \| \sum_{k \in P \setminus Q} \psi_k\| \leq D\| \sum_{k \in Q \setminus P} \psi_k\|.$$

Combining (2.6)–(2.12) we obtain (2.5). Theorem 2.2 is proved.

We now discuss a stability of greedy type property of a basis. Let $0 < a \leq \lambda_k \leq b < \infty$, $k = 1, 2, \ldots$ and for a basis $\Psi = \{\psi_k\}$ consider $\Psi^\lambda := \{\lambda_k \psi_k\}$.

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Theorem 2.3. *Let a basis Ψ have one of the properties*
  1. Greedy.
  2. Almost greedy.
  3. Quasi-greedy.
  Then the basis Ψ^λ has the same property.

Proof. Let $f \in X$ and

$$f = \sum_k c_k(f)\psi_k = \sum_k c_k(f)\lambda_k^{-1}\lambda_k\psi_k.$$

Consider

$$G_m(f, \Psi^\lambda) = \sum_{k \in \Lambda_m} (c_k(f)\lambda_k^{-1})\lambda_k\psi_k.$$

Then we have

$$\min_{k \in \Lambda_m} |c_k(f)| \ge a \min_{k \in \Lambda_m} |c_k(f)|\lambda_k^{-1} \ge a \max_{k \notin \Lambda_m} |c_k(f)|\lambda_k^{-1} \ge \frac{a}{b} \max_{k \notin \Lambda_m} |c_k(f)|.$$

Therefore, the set Λ_m can be interpreted as a $\Lambda_m(t)$ with $t = a/b$ with regard to the basis Ψ. It remains to apply the corresponding results for $G_m^t(f, \Psi)$: (2.3) in the case 1, Theorem 2.2 in the case 2, and Theorem 2.1 in the case 3. This completes the proof of Theorem 2.3.

In the paper [KaT] the following modification of the above weak type greedy algorithm in a way of further weakening the restriction (2.1) has been studied. We call this modification the Weak Thresholding Greedy Algorithm (WTGA) with a weakness sequence $\tau = \{t_k\}$. Let a weakness sequence $\tau := \{t_k\}_{k=1}^\infty$, $t_k \in [0,1]$, $k = 1, \ldots$ be given. We define the WTGA by induction. We take an element $f \in X$ and at the first step we let

$$\Lambda_1(\tau) := \{n_1\}; \qquad G_1^\tau(f, \Psi) := c_{n_1}\psi_{n_1}$$

with n_1 any satisfying

$$|c_{n_1}| \ge t_1 \max_n |c_n|$$

where we denote for brevity $c_n := c_n(f, \Psi)$. Assume we have already defined

$$G_{m-1}^\tau(f, \Psi) := G_{m-1}^{X,\tau}(f, \Psi) := \sum_{n \in \Lambda_{m-1}(\tau)} c_n\psi_n.$$

Then at the mth step we define

$$\Lambda_m(\tau) := \Lambda_{m-1}(\tau) \cup \{n_m\}; \qquad G_m^\tau(f, \Psi) := G_m^{X,\tau}(f, \Psi) := \sum_{n \in \Lambda_m(\tau)} c_n\psi_n$$

with $n_m \notin \Lambda_{m-1}(\tau)$ any satisfying

$$|c_{n_m}| \ge t_m \max_{n \notin \Lambda_{m-1}(\tau)} |c_n|.$$

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Thus for an $f \in X$ the WTGA builds a rearrangement of a subsequence of the expansion (1.1). If Ψ is an unconditional basis then always $G_m^\tau(f, \Psi) \to f^*$. It is clear that in this case $f^* = f$ if and only if the sequence $\{n_k\}_{k=1}^\infty$ contains indices of all nonzero $c_n(f, \Psi)$. We say that the WTGA corresponding to Ψ and τ is convergent (converges) if for any realization $G_m^\tau(f, \Psi)$ we have

$$\|f - G_m^\tau(f, \Psi)\| \to 0 \quad \text{as} \quad m \to \infty$$

for all $f \in X$.

In [KaT] the following three theorems on convergence of the WTGA have been proved. The first one deals with an arbitrary Banach space X and any basis Ψ.

Theorem 2.4. *Let X be a Banach space with a normalized basis Ψ. Let $\tau = \{t_n, n \geq 1\}$ be a weakness sequence. The following condition (D) is a necessary condition for the WTGA corresponding to Ψ and τ to be convergent.*

(D) For each subsequence $\{n_k, k \geq 1\}$ of different indices, the series $\sum_{k=1}^\infty t_k \psi_{n_k}$ diverges in X.

If the basis Ψ is unconditional, then the above condition (D) is also a sufficient condition for the WTGA corresponding to Ψ and τ to be convergent.

In the case $X = L_p([0,1]^d)$ one can derive from Theorem 2.4 a more specific condition in terms of τ (see [KaT]).

Theorem 2.5. *Let $2 \leq p < \infty$, $d \geq 1$ and let Ψ be a normalized unconditional basis in $L_p([0,1]^d)$. Let $\tau = \{t_n, n \geq 1\}$ be a weakness sequence. Then the WTGA corresponding to Ψ and τ converges if and only if $\tau \notin l_p$.*

There is no simple criterion in terms of τ in the case $X = L_p([0,1]^d)$, $1 < p < 2$ and arbitrary unconditional basis Ψ. In this case [KaT] contains the following result for the multivariate Haar basis $\mathcal{H}_p^d$ defined as the tensor product of the univariare Haar bases: $\mathcal{H}_p^d := \mathcal{H}_p \times \cdots \times \mathcal{H}_p$. To formulate the result, introduce the following notation. For a sequence $\{t_k, k \geq 1\}$ of nonnegative numbers such that $\lim_{k\to\infty} t_k = 0$, $\{t_k^*, k \geq 1\}$ is a nonincreasing rearrangement of the subsequence $\{t_{n_k}, k \geq 1\}$ consisting of positive elements of $\{t_k, k \geq 1\}$.

Theorem 2.6. *Let $d \geq 1$ and $1 < p < 2$. The WTGA corresponding to $\mathcal{H}_p^d$ and a weakness sequence τ converges in $L_p([0,1]^d)$ if and only if one of the following conditions is satisfied:*
(i) The sequence $\tau = \{t_k\}$ does not converge to 0.
(ii) $\lim_{k\to\infty} t_k = 0$ and

(2.13)
$$\sum_{k=1}^\infty (t_k^*)^2 \big(k(\log k)^{(1-d)}\big)^{2/p-1} = \infty.$$

Along with convergence of the WTGA efficiency of approximation by $G_m^\tau(\cdot, \Psi)$ has been studied in [KaT]. The accuracy of the WTGA was compared with best m-term approximation. In the case of greedy basis and $\tau = \{t\}$, $t \in (0,1]$ the relation (2.3) shows that $G_m^\tau(\cdot, \Psi)$ realizes near best m-term approximation. There are two natural ways of adapting

(2.3) to the case of nongreedy basis or to the case of general weakness sequence. In the first way (see [T5], [T3], [W], [Os]) we write (2.3) in the form

$$\|f - G_m^\tau(f, \Psi)\| \le C(m, \tau, \Psi)\sigma_m(f, \Psi)$$

and look for the best (in the sense of order) constant $C(m, \tau, \Psi)$.

We now formulate the correspoding results from [KaT]. For a basis Ψ we define the fundamental function $\varphi(m)$ and the function $\phi(m)$ like in Section 1. We also need the following function

$$\varphi^s(m) := \sup_{|A|=m} \|\sum_{k \in A} \psi_k\|.$$

It is clear that

$$\varphi(m) = \sup_{n \le m} \varphi^s(n).$$

We now introduce some characteristics of a basis with respect to a weakness sequence τ. For a subset $V \subseteq [1, m]$ of integers we define

$$\phi(\tau, m, V) := \inf_{\{k_i\}} \|\sum_{i \in V} t_i \psi_{k_i}\|$$

where inf is taken over all sets $\{k_i\}$ of different indices. For two integers $1 \le n \le m$ we define

$$\phi(\tau, m, n) := \inf_{|V|=n, V \subseteq [1,m]} \phi(\tau, m, V),$$

and finally

$$\mu(\tau, m) := \sup_{n \le m} \frac{\varphi^s(n)}{\phi(\tau, m, n)}.$$

The following result has been proved in [KaT].

Theorem 2.7. *Let Ψ be a normalized unconditional basis for X. Then we have*

$$\|f - G_m^\tau(f, \Psi)\| \le C(\Psi)\mu(\tau, m)\sigma_m(f, \Psi).$$

In Theorem 2.7 we compare efficiency of $G_m^\tau(\cdot, \Psi)$ with $\sigma_m(\cdot, \Psi)$. It is known in approximation theory that sometimes it is convenient to compare efficiency of an approximating operator which is characterized by m parameters with best possible approximation corresponding to smaller number of parameters $n \le m$. We use this idea in approximation by the WTGA. Let us discuss a setting (see [KaT]) when we write (2.3) in the form

$$\|f - G_{v_m}^\tau(f, \Psi)\| \le C(\Psi)\sigma_m(f, \Psi)$$

and look for the best (in the sense of order) sequence $\{v_m\}$ that is determined by the weakness sequence τ and the basis Ψ. We need some more notation. Define

$$\phi(\tau, N) := \phi(\tau, N, [1, N]) = \inf_{k_1, \ldots, k_N} \|\sum_{j=1}^N t_j \psi_{k_j}\|.$$

Assume that $\phi(\tau, N) \to \infty$ as $N \to \infty$ and denote v_m the smallest N satisfying

$$\phi(\tau, N) \ge 2\varphi(m).$$

There is the following result ([KaT]) in this case.

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Theorem 2.8. *For any normalized unconditional basis Ψ we have*

$$\|f - G^\tau_{v_m}(f, \Psi)\| \le C(\Psi)\sigma_m(f, \Psi).$$

It is interesting to compare this result with some recent results from [DKKT]. It has been established in [DKKT] (see Theorem 1.4 of present paper) that the inequalities

$$(2.14) \qquad \qquad \|f - G_{[\lambda m]}(f, \Psi)\| \le C(\Psi, \lambda)\sigma_m(f, \Psi)$$

with fixed $\lambda > 1$ are characteristic for a class of almost greedy bases. It is clear that each greedy basis is an almost greedy basis. There is an example (see [KT1, sections 3.3, 3.4]) of almost greedy basis that is not a greedy basis. This means that $\lambda > 1$ needed for (2.14) can not be replaced by $\lambda \ge 1$.

3. Thresholding type approximation with regard to minimal systems

Let X be a quasi-Banach space (real or complex) with the quasi-norm $\|\cdot\|$ such that for all $x, y \in X$ we have $\|x + y\| \le \alpha(\|x\| + \|y\|)$ and $\|tx\| = |t|\|x\|$. It is well-known (see [KBR, Lemma 1.1]) that there is a p, $0 < p \le 1$, such that

$$(3.1) \qquad \qquad \left\|\sum_n x_n\right\| \le 4^{1/p}\left(\sum_n \|x_n\|^p\right)^{1/p}.$$

Let $\{e_n\} \subset X$ be a complete minimal system in X with the conjugate (dual) system $\{e_n^*\} \subset X^*$. We assume that $\sup_n \|e_n^*\| < \infty$. This implies that for each $x \in X$ we have

$$(3.2) \qquad \qquad \lim_{n \to \infty} e_n^*(x) = 0.$$

Any element $x \in X$ has a formal expansion

$$(3.3) \qquad \qquad x \sim \sum_n e_n^*(x)e_n,$$

and various types of convergence of the series (3.3) can be studied. In this section we deal with greedy type approximations with regard to the system $\{e_n\}$. We note that in this section we use the notations x and $\{e_n\}$ for an element and for a system respectively different from the notations f and Ψ in the previous sections to emphasize that we are in a more general setting now. It will be convenient for us to define a unique "greedy ordering" in this section. For any $x \in X$ we define the greedy ordering for x as the map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\{j : e_j^*(x) \ne 0\} \subset \rho(\mathbb{N})$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. The m-th greedy approximation is given by

$$G_m(x) := G_m(x, \{e_n\}) := \sum_{j=1}^m e_{\rho(j)}^*(x)e_{\rho(j)}.$$

The system $\{e_n\}$ is called a quasi-greedy system (see [KT1]) if there exists a constant C such that $\|G_m(x)\| \leq C\|x\|$ for all $x \in X$ and $m \in \mathbb{N}$. Wojtaszchyk [W] proved that these are precisely the systems for which $\lim_{m\to\infty} G_m(x) = x$ for all x. If a quasi-greedy system $\{e_n\}$ is a basis then we say that $\{e_n\}$ is a quasi-greedy basis. It is clear that any unconditional basis is a quasi-greedy basis. We note that there are conditional quasi-greedy bases $\{e_n\}$ in some Banach spaces [KT, W]. Hence, for such a basis $\{e_n\}$ there exists a permutation of $\{e_n\}$ which forms a quasi-greedy system but not a basis. This remark justifies the study of the class of quasi-greedy systems rather than the class of quasi-greedy bases.

Greedy approximations are close to thresholding approximations (sometimes they are called "thresholding greedy approximations"). Thresholding approximations are defined as

$$T_\epsilon(x) = \sum_{|e_j^*(x)| \geq \varepsilon} e_j^*(x)e_j, \quad \epsilon > 0.$$

Clearly, for any $\varepsilon > 0$ there exists an m such that $T_\epsilon(x) = G_m(x)$. Therefore, if $\{e_n\}$ is a quasi-greedy system then

$$(3.4) \qquad\qquad \forall x \in X \quad \lim_{\varepsilon \to 0} T_\epsilon(x) = x.$$

Conversely, following Remark from [W, pages 296–297], it is easy to show that the condition (3.4) implies that $\{e_n\}$ is a quasi-greedy system.

Similarly to the above, one can define the Weak Thresholding Approximation. Fix $t \in (0,1)$. For $\varepsilon > 0$ denote

$$D_{t,\varepsilon}(x) := \{j : t\varepsilon \leq |e_j^*(x)| < \varepsilon\}.$$

The Weak Thresholding Approximations are defined as all possible sums

$$T_{\epsilon,D}(x) = \sum_{|e_j^*(x)| \geq \varepsilon} e_j^*(x)e_j + \sum_{j \in D} e_j^*(x)e_j,$$

where $D \subseteq D_{t,\varepsilon}(x)$. We say that the Weak Thresholding Algorithm converges for $x \in X$ and write $x \in WT\{e_n\}(t)$ if for any $D(\epsilon) \subseteq D_{t,\epsilon}$

$$\lim_{\varepsilon \to 0} T_{\epsilon,D(\epsilon)}(x) = x.$$

It is clear that the above relation is equivalent to

$$\lim_{\varepsilon \to 0} \sup_{D \subseteq D_{t,\varepsilon}(x)} \|x - T_{\epsilon,D}(x)\| = 0.$$

We proved in [KT2] (see Theorem 3.1 below) that the set $WT\{e_n\}(t)$ does not depend on t. Therefore, we can drop t from the notation: $WT\{e_n\} = WT\{e_n\}(t)$.

It turns out that the Weak Thresholding Algorithm has more regularity than the Thresholding Algorithm: we will see that the set $WT\{e_n\}$ is linear. On the other hand, by "weakening" the Thresholding Algorithm (making convergence stronger) we do not narrow the convergence set too much. It is known that for many natural classes of sets $Y \subseteq X$ the convergence of $T_\epsilon(x)$ to x for all $x \in Y$ is equivalent to the condition $Y \subseteq WT\{e_n\}$. In particular, it can be derived from [W, Proposition 3] that the two above conditions are equivalent for $Y = X$.

We suppose that X and $\{e_n\}$ satisfy the conditions stated in the beginning of this section. The following two theorems have been proved in [KT2].

Theorem 3.1. *Let $t, t' \in (0,1)$, $x \in X$. Then the following conditions are equivalent:*

1) $\lim_{\varepsilon \to 0} \sup_{D \subseteq D_{t,\varepsilon}(x)} \|T_{\epsilon, D}(x) - x\| = 0$;

2) $\lim_{\varepsilon \to 0} T_{\epsilon}(x) = x$ *and*

$$(3.5) \qquad \lim_{\varepsilon \to 0} \sup_{D \subseteq D_{t,\varepsilon}(x)} \| \sum_{j \in D} e_j^*(x) e_j \| = 0;$$

3) $\lim_{\varepsilon \to 0} T_{\epsilon}(x) = x$ *and*

$$(3.6) \qquad \lim_{\varepsilon \to 0} \sup_{|a_j| \leq 1 (j \in D_{t,\varepsilon}(x))} \| \sum_{j \in D_{t,\varepsilon}(x)} a_j e_j^*(x) e_j \| = 0;$$

4) $\lim_{\varepsilon \to 0} T_{\epsilon}(x) = x$ *and*

$$(3.7) \qquad \lim_{\varepsilon \to 0} \sup_{|b_j| < \varepsilon (j : |e_j^*(x)| \geq \varepsilon)} \| \sum_{j : |e_j^*(x)| \geq \varepsilon} b_j e_j \| = 0;$$

5) $\lim_{\varepsilon \to 0} \sup_{D \subseteq D_{t',\varepsilon}(x)} \|T_{\epsilon, D}(x) - x\| = 0$.

So, the set $WT\{e_n\}(t)$ defined above is indeed independent of $t \in (0,1)$.

Theorem 3.2. *The set $WT\{e_n\}$ is linear.*

Let us discuss relations between the Weak Thresholding Algorithm $T_{\epsilon, D}(x)$ and the Weak Thresholding Greedy Algorithm $G_m^t(x)$. We define $G_m^t(x)$ with regard to a minimal system $\{e_n\}$ in the same way as it was defined for a basis Ψ. For a given system $\{e_n\}$ and $t \in (0,1]$ we denote for $x \in X$ and $m \in \mathbb{N}$ by $W_m(t)$ any set of m indices such that

$$(3.8) \qquad \min_{j \in W_m(t)} |e_j^*(x)| \geq t \max_{j \notin W_m(t)} |e_j^*(x)|$$

and define

$$G_m^t(x) := G_m^t(x, \{e_n\}) := S_{W_m(t)}(x) := \sum_{j \in W_m(t)} e_j^*(x) e_j.$$

It is clear that for any $t \in (0,1]$ and any $D \subseteq D_{t,\epsilon}(x)$ there exist m and $W_m(t)$ satisfying (3.8) such that

$$T_{\epsilon, D}(x) = S_{W_m(t)}(x).$$

Thus the convergence $G_m^t(x) \to x$ as $m \to \infty$ implies the convergence $T_{\epsilon, D}(x) \to x$ as $\epsilon \to \infty$ for any $t \in (0,1]$. We will now prove (see [KT2], Proposition 2.2) that for $t \in (0,1)$ the inverse is also true.

Proposition 3.1. *Let $t \in (0,1)$ and $x \in X$. Then the following two conditions are equivalent:*

$$(3.9) \qquad \lim_{\epsilon \to 0} \sup_{D \subseteq D_{t,\epsilon}(x)} \|T_{\epsilon, D}(x) - x\| = 0;$$

(3.10)
$$\lim_{m \to \infty} \|G_m^t(x) - x\| = 0$$

for any realization $G_m^t(x)$.

Proof. The implication (3.10) $\Rightarrow$ (3.9) is simple and follows from a remark preceding Proposition 3.1. We prove that (3.9) $\Rightarrow$ (3.10). Denote

$$\epsilon_m := \max_{j \notin W_m(t)} |e_j^*(x)|.$$

Clearly $\epsilon_m \to 0$ as $m \to \infty$. We have

(3.11)
$$G_m^t(x) = T_{2\epsilon_m}(x) + \sum_{j \in D_m} e_j^*(x) e_j$$

with D_m having the following property: for any $j \in D_m$

$$t\epsilon_m \le |e_j^*(x)| < 2\epsilon_m.$$

Thus by condition 5) from Theorem 3.1 for $t' = t/2$ we obtain (3.11).
 Proposition 3.1 is now proved.

 Proposition 3.1 and Theorem 3.1 imply that the convergence set of the Weak Thresholding Greedy Algorithm $G_m^t(\cdot)$ does not depend on $t \in (0,1)$ and coincides with $WT\{e_n\}$. By Theorem 3.2 this set is a linear set.
 Let us make a comment on the case $t = 1$ that is not covered by Proposition 3.1. It is clear that $T_\epsilon(x) = G_m(x)$ with some m and, therefore, $G_m(x) \to x$ as $m \to \infty$ implies $T_\epsilon(x) \to x$ as $\epsilon \to 0$. It is also not difficult to understand that in general $T_\epsilon(x) \to x$ as $\epsilon \to 0$ does not imply $G_m(x) \to x$ as $m \to \infty$. This can be done, for instance, considering the trigonometric system in the space L_p, $p \neq 2$, and using the Rudin-Shapiro polynomials (see [T5]). However, if for the trigonometric system we put the Fourier coefficients with equal absolute values in a natural order (say, lexicographic), then in the case $1 < p < \infty$ by Riesz theorem we obtain convergence of $G_m(f)$ from convergence of $T_\epsilon(f)$. Results from the paper [KS] show that the situation is different for $p = 1$. In this case the natural order does not help to derive convergence of $G_m(f)$ from convergence of $T_\epsilon(f)$.
 Let us give an application of the results of this section for summation of number series. A series $\sum_n a_n$, $a_n \in \mathbb{C}$, is said to A-converge to a number $s \in \mathbb{C}$ if the following conditions hold:

(3.12)
$$\lim_{\epsilon \to 0^+} \sum_{n:|a_n| \ge \varepsilon} a_n = s;$$

(3.13)
$$\lim_{\epsilon \to 0^+} \varepsilon |\{n : |a_n| \ge \varepsilon\}| = 0.$$

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We shall write it as

$$(A) \sum_n a_n = s.$$

The notion of A-convergent series has been studied in [U2]; see also [U3]. It is similar to the well-known notion of the A-integral (see, e.g., [U1]). We show that A-convergence can be treated as weak thresholding convergence of number series. Recall that c_0 is the space of sequences convergent to zero. Namely,

$$c_0 = \left\{ x = (x^0, x^1, \dots) : x^n \in \mathbb{C}, \ \lim_{n \to \infty} x^n = 0 \right\},$$

with the norm of $x \in c_0$ defined as $\|x\| = \max_n |x_n|$. It is known that

$$c_0^* = l_1 = \left\{ (x^0, x^1, \dots) : x^n \in \mathbb{C}, \ \|x\| = \sum_{n=0}^{\infty} |x^n| < \infty \right\}.$$

Consider the system $\{e_n\}_{n \in \mathbb{N}} \subset c_0$ defined as $e_n^0 = e_n^n = 1$, $e_n^j = 0$ for $j \neq 0, n$. It is clear that $\{e_n\}$ is a minimal system. It is also easy to see that $\{e_n\}$ is complete in c_0. For instance, we have for the coordinate vectors u_n $(u_n^n = 1, u_n^j = 0, j \neq n)$, $n = 0, 1, \dots$:

$$\left\| u_0 - \frac{1}{m} \sum_{n=1}^{m} e_n \right\|_{c_0} \leq 1/m;$$

$$u_n = e_n - u_0, \quad n = 1, 2, \dots.$$

The elements e_n^* of the conjugate system are $e_n^* = u_n$, $n = 1, 2, \dots$. Thus, the formal expansion (3.3) takes the form

$$x \sim \sum_{n=1}^{\infty} x^n e_n.$$

Clearly, this expansion converges to x for $x \in c_0$ satisfying the following condition

$$x^0 = \sum_{n=1}^{\infty} x^n.$$

Theorem 3.3. *Define the system $\{e_n\}_{n \in \mathbb{N}} \subset c_0$ as $e_n^0 = e_n^n = 1$, $e_n^j = 0$ for $j \neq 0, n$. Let $\sum_{n \in \mathbb{N}} a_n$ be a number series, $\lim_{n \to \infty} a_n = 0$, $s \in \mathbb{C}$, $t \in (0, 1)$. Then the following conditions are equivalent:*
1) the series $\sum_n a_n$ A-converges to s;
2) $\lim_{\varepsilon \to 0} \sup_{D \subseteq D_{t, \varepsilon}} |T_{\epsilon, D} - s| = 0$, where

$$D_{t, \varepsilon} = \{ j : t\varepsilon \leq |a_j| < \varepsilon \}, \quad T_{\epsilon, D} = \sum_{|a_j| \geq \varepsilon} a_j + \sum_{j \in D} a_j;$$

3) the element $x \in c_0$ defined as $x = (s, a_1, a_2, \dots)$ belongs to $WT\{e_n\}$.

The following corollary of Theorems 3.2 and 3.3 has been proved in [U2].

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Corollary 3.1. *The set of A-convergent series is linear. Moreover,*

$$(A) \sum_n (a_n + b_n) = (A) \sum_n a_n + (A) \sum_n b_n.$$

We have already made some remarks justifying consideration of minimal systems instead of bases in the study of greedy type algorithms. We will make a remark (see [KT2]) showing that the step from Banach spaces to quasi-Banach spaces is also natural in studying greedy type algorithms.

Remark 3.1. *One can check (see the proof of Theorem 3.1 in [KT2]) that for any $t \in (0,1)$ the quasi-norm $\|\|\cdot\|\|_t$ in the space $Y = WT\{e_n\} \in c_0$ defined as*

$$\|\|x\|\|_t = \sup_\varepsilon \sup_{D \subseteq D_{t,\varepsilon}(x)} \|T_{\epsilon,D}(x)\|$$

is equivalent to the quasi-norm

$$\|\|x\|\| = \max(|x^0|, \sup_\varepsilon \varepsilon |\{n \geq 1 : |x^n| \geq \varepsilon\}|).$$

Also, a quasi-norm in the space Y can be treated as a quasi-norm in the space of A-convergent series.

Theorem 3.4. *The quasi-norm $\|\|\cdot\|\|$ in the space $Y = WT\{e_n\} \in c_0$ is not equivalent to any norm.*

Proof. It is sufficient to show that for any $M > 0$ there exist a positive integer m and elements $x_1, \ldots, x_m$ from Y such that

(3.14) $$\|\|x_j\|\| \leq 1 \quad (j = 1, \ldots, m)$$

and

(3.15) $$\left\|\left\| \frac{1}{m} \sum_{j=1}^m x_j \right\|\right\| > M.$$

Take an even $m \in \mathbb{N}$ and set $x_j^n = 0$ for $n > m$, $x_j^n = (-1)^n/k$ for $1 \leq n \leq m$ where $k \in \{1, \ldots, m\}$ is defined as $k \equiv n + j (\mathrm{mod} m)$, $x_j^0 = \sum_{n=1}^m x_j^n$. It is easy to see that all the elements $x_j = (x_j^0, x_j^1, \ldots)$ satisfy (3.14). Further, for the element $x = \frac{1}{m} \sum_{j=1}^m x_j = (x^0, x^1, \ldots)$ we have

$$|x^n| = \frac{1}{m} \sum_{k=1}^m 1/k \quad (n = 1, \ldots, m).$$

Therefore, $\|\|x\|\| \geq \sum_{k=1}^m 1/k$, and (3.15) holds for sufficiently large m. The proof of Theorem 3.4 is complete.

The reader can also find in [KT2, S.4] applications of these results for studying A-convergence of trigonometric series.

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References

[CDH]   A. Cohen, R.A. DeVore, and R. Hochmuth, *Restricted Nonlinear Approximation*, Constructive Approx. **16** (2000), 85–113.

[CF]   A. Cordoba and P. Fernandez, *Convergence and divergence of decreasing rearranged Fourier series*, SIAM, I. Math. Anal. **29** (1998), 1129–1139.

[D]   R.A. DeVore, *Nonlinear approximation*, Acta Numerica (1998), 51–150.

[DJP]   R. DeVore, B. Jawerth, and V. Popov, *Compression of wavelet decompositions*, American Journal of Mathematics **114** (1992), 737–785.

[DKT]   R.A. DeVore, S.V. Konyagin, and V.N. Temlyakov, *Hyperbolic wavelet approximation*, Constructive Approximation **14** (1998), 1–26.

[DKKT]   S.J. Dilworth, N.J. Kalton, D. Kutzarova, V.N. Temlyakov, *The Tresholding Greedy Algorithm, Greedy Bases, and Duality*, IMI-Preprints series **23** (2001), 1–23.

[DM]   S.J. Dilworth and D. Mitra, *A conditional quasi-greedy basis of L_1*, Studia Math **144** (2001), 95–100.

[DT]   R.A. DeVore and V.N. Temlyakov, *Some remarks on Greedy Algorithms*, Advances in Computational Mathematics **5** (1996), 173–187.

[FJ]   M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Functional Analysis **93** (1990), 34-170.

[GM]   R. Gribonval and M. Nielsen, *Some remarks on non-linear approximation with Schauder bases*, East J. Approx. **7** (2001), 267–285.

[KaT]   A. Kamont and V.N. Temlyakov, *Greedy approximation and the multivariate Haar system*, manuscript (2002), 1–23.

[KBR]   N.J. Kalton, N.T. Beck, and J.W. Roberts, *An F-space sampler*, London Math. Soc. Lecture Notes, Cambridge Univ. Press, Cambridge. UK **5** (1984).

[KP]   G. Kerkyacharian and D. Picard, *Entropy, universal coding, approximation and bases properties*, University of Paris 6 and 7, Preprint **663** (2001), 1–32.

[KS]   S.V. Konyagin and M.A. Skopina, *Comparison of the L_1-norms of total and truncated exponential sums*, Mat. Zametki **69** (2001), 699–707.

[KT1]   S.V. Konyagin and V.N. Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. on Approx. **5** (1999), 1–15.

[KT2]   S.V. Konyagin and V.N. Temlyakov, *Convergence of Greedy Approximation I. General Systems*, IMI-Preprint series **08** (2002), 1–19.

[LT]   J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin, 1979.

[Os]   P. Oswald, *Greedy algorithms and best m-term approximation with respect to biorthogonal systems*, Preprint (2000), 1–22.

[St]   S. B. Stechkin, *On absolute convergence of orthogonal series*, Dok. Akad. Nauk SSSR **102** (1955), 37-40.

[T1]   V.N. Temlyakov, *Some inequalities for multivariate Haar polynomials*, East J. Approx. **1** (1995), 61–72.

[T2]   V.N. Temlyakov, *The best m-term approximation and Greedy Algorithms*, Advances in Comp. Math. **8** (1998), 249–265.

[T3]   V.N. Temlyakov, *Nonlinear m-term approximation with regard to the multivariate Haar system*, East J. Approx. **4** (1998), 87–106.

[T4]   V.N. Temlyakov, *Greedy algorithms with regard to multivariate systems with special structure*, Constructive Approximation **16** (2000), 399–425.

[T5]   V.N. Temlyakov, *Greedy Algorithm and m-term Trigonometric Approximation*, Constructive Approx. **14** (1998), 569–587.

[T6]   V.N. Temlyakov, *Universal bases and greedy algorithms*, IMI-Preprint series **8** (1999), 1–20.

[T7]   V.N. Temlyakov, *Nonlinear approximation with regard to bases*, Approximation Theory X, Vanderbilt University Press, Nashville, TN, 373–402.

19

[Te1]   V. Temlyakov, *Approximation of functions with bounded mixed derivative*, Proc. Steklov Institute (1989, Issue 1).

[Te2]   V.N. Temlyakov, *Approximation of periodic functions*, Nova Science Publishers, Inc., New York, 1993.

[U1]   P. L. Ul'yanov, *Integrals of Cauchy type*, Tr. Mat. Inst. im. Steklova **60** (1961), 262–281.

[U2]   P. L. Ul'yanov, *On the A-convergence of series*, Dokl. Math. **60** (1999), 176–179.

[U3]   P. L. Ul'yanov, *On the properties of series*, Dokl. Math. **62** (2000), 56–59.

[W]   P. Wojtaszczyk, *Greedy algorithms for general systems*, J. Approx. Theory **107** (2000), 293–314.