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S.J. Dilworth, N.J. Kalton, D. Kutzarova and V.N. Temlyakov

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THE THRESHOLDING GREEDY ALGORITHM, GREEDY BASES, AND DUALITY

S. J. DILWORTH, N. J. KALTON, DENKA KUTZAROVA, AND V. N. TEMLYAKOV

Abstract. Some new conditions that arise naturally in the study of the Thresholding Greedy Algorithm are introduced for bases of Banach spaces. We relate these conditions to best n-term approximation and we study their duality theory. In particular, we obtain a complete duality theory for greedy bases.

Contents

1. Introduction

Let X be a Banach space with a basis (e_n). An approximation algorithm $(F_n)_{n=1}^{\infty}$ is a sequence of maps $F_n : X \to X$ such that for each $x \in X$, $F_n(x)$ is a linear combination of at most n of the basis elements (e_j). The most natural algorithm is the *linear algorithm* $(S_n)_{n=1}^{\infty}$ given by the partial sum operators.

Recently, Konyagin and Temlyakov [5] introduced the *Thresholding Greedy Algorithm* (TGA) $(G_n)_{n=1}^{\infty}$, where $G_n(x)$ is obtained by taking the largest n coefficients (precise definitions are given in Section 2). The TGA provides a theoretical model for the thresholding procedure that is used in image compression and other applications.

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They defined the basis (e_n) to be *greedy* if the TGA is optimal in the sense that $G_n(x)$ is essentially the best n-term approximation to x using the basis vectors, i.e. there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$ we have

$$\|x - G_n(x)\| \leq C \inf\{\|x - \sum_{j\in A} \alpha_j e_j\| : \ |A| = n, \ \alpha_j \in \mathbb{R}, \ j \in A\}. \tag{1.1}$$

They then showed that greedy bases can be simply characterized as unconditional bases with the additional property of being *democratic*, i.e. for some $\Delta > 0$ we have $\|\sum_{j\in A} e_j\| \leq \Delta \|\sum_{j\in B} e_j\|$ whenever $|A| \leq |B|$.

They also defined a basis to be *quasi-greedy* if there exists a constant C such that $\|G_m(x)\| \leq C\|x\|$ for all $x \in X$ and $n \in \mathbb{N}$. Subsequently, Wojtaszczyk [9] proved that these are precisely the bases for which the TGA merely converges, i.e. $\lim_{n\to\infty} G_n(x) = x$ for $x \in X$.

In this paper we introduce two natural intermediate conditions. Let us denote the biorthogonal sequence by (e_n^*). We say (e_n) is *almost greedy* if there is a constant C such that

$$\|x - G_n(x)\| \leq C \inf\{\|x - \sum_{j\in A} e_j^*(x) e_j\| : \ |A| = n\} \qquad x \in X, \ n \in \mathbb{N}. \tag{1.2}$$

Comparison with (1.1) shows that this is formally a weaker condition; in fact Wojtaszczyk's examples of conditional quasi-greedy bases of ℓ_2 [9] are almost greedy but not greedy. We give two characterizations of almost greedy bases in Theorem 3.3. First, a basis is almost greedy if and only if it is quasi-greedy and democratic. Second, if $\lambda > 1$, then $(e_n)_{n=1}^\infty$ is almost greedy if and only if there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$, we have

$$\|x - G_{[\lambda n]}(x)\| \leq C \inf\{\|x - \sum_{j\in A} \alpha_j e_j\| : \ |A| = n, \ \alpha_j \in \mathbb{R}, \ j \in A\}. \tag{1.3}$$

Equation (1.2) is a very natural weakening of (1.1).

We also introduce *partially greedy* bases. These are bases such that for some C we have

$$\|x - G_n(x)\| \leq C\|\sum_{k=n+1}^{\infty} e_k^*(x) e_k\| \quad x \in X, n \in \mathbb{N}. \tag{1.4}$$

We give a characterization in Theorem 3.4.

Next we study duality of these conditions. In Theorem 5.1 we show that if (e_n) is a greedy basis of a Banach space X with nontrivial Rademacher type then (e_n^*) is a greedy basis of X^*. However, examples at the end of the paper show that if X does not have type then (e_n^*) need not be a greedy basic sequence. Theorem 5.4 generalizes Theorem 5.1 by showing that if (e_n) is any quasi-greedy basis then (e_n) and (e_n^*) are both partially greedy basic sequences if and only if they are both almost greedy basic sequences if and only if (e_n) is *bidemocratic*, i.e. for some C we have

$$\|\sum_{j\in A} e_j\|\|\sum_{j\in A} e_j^*\| \leq Cn \quad |A| = n,\ n \in \mathbb{N}.$$

Using this result we extend Theorem 5.1 by showing that if X has nontrivial type and (e_n) is almost greedy then (e_n^*) is an almost greedy basic sequence.

We use standard Banach space notation throughout (see e.g. [7]). For clarity, however, we recall here the notation that is used most heavily. Let X be a Banach space. The *dual space* of X, denoted X^*, is the Banach space of all continuous linear functionals F equipped with the norm:

$$\|F\| = \sup\{F(x)\colon \|x\| = 1\}.$$

The closed linear span of a set $A \subseteq X$ (resp., a sequence (x_n)) is denoted $[A]$ (resp. $[x_n]$). A *basis* for X is a sequence of vectors (e_n) such that every $x \in X$ has a unique expansion as a norm-convergent series

$$x = \sum_{k=1}^{\infty} e_n^*(x) e_n.$$

Here (e_n^*) is the sequence of *biorthogonal functionals* in X^* defined by $e_n^*(e_m) = \delta_{n,m}$. The basis is said to be *unconditional* if the series expansion converges unconditionally for every $x \in X$. It is said to be *monotone* if

$$\|\sum_{k=1}^{n} e_k^*(x) e_k\| \leq \|x\| \qquad (x \in X, n \geq 1).$$

Finally, more specialized notions from Banach space theory, such as *type* and *cotype*, will be introduced as needed.

2. Greedy conditions for bases

Let $(e_n)_{n\in\mathbb{N}}$ be a basis of a Banach space X; let $(e_n^*)_{n\in\mathbb{N}}$ be the biorthogonal sequence in X^*. Let us denote by S_m the partial-sum operators:

$$S_m(x) = \sum_{j=1}^{m} e_j^*(x) e_j.$$

We also define the remainder operators $R_m = I - S_m$. For any $x \in X$ we define the *greedy ordering for* x as the map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\rho(\mathbb{N}) \supset \{j : e_j^*(x) \neq 0\}$ and so that if $j < k$ then either $|e^*_{\rho(j)}(x)| > |e^*_{\rho(k)}(x)|$ or $|e^*_{\rho(j)}(x)| = |e^*_{\rho(k)}(x)|$ and $\rho(j) < \rho(k)$. The m-th greedy approximation is given by

$$G_m(x) = \sum_{j=1}^{m} e^*_{\rho(j)}(x) e_{\rho(j)}.$$

We will also introduce the m-th greedy remainder

$$H_m(x) = x - G_m(x).$$

The basis (e_n) is called *quasi-greedy* if $G_m(x) \to x$ for all $x \in X$. This is equivalent (see [9]) to the condition that for some constant C we have

$$\sup_m \|G_m(x)\| \le C\|x\| \qquad x \in X. \tag{2.1}$$

It will be convenient to define the *quasi-greedy constant* K to be the least constant such that

$$\|G_m(x)\| \le K\|x\| \quad \text{and} \quad \|H_m(x)\| \le K\|x\| \quad x \in X.$$

If (e_n) is any basis we denote

$$\sigma_m(x) = \inf\{\|x - \sum_{j\in A} \alpha_j e_j\| : \ |A| = m, \ \alpha_j \in \mathbb{R}\}.$$

A basis (e_n) is called *greedy* [5] if there is a constant C such that for any $x \in X$ and $m \in \mathbb{N}$ we have

$$\|H_m(x)\| \le C\sigma_m(x). \tag{2.2}$$

It is natural to introduce two slightly weaker forms of greediness. For any basis (e_n) let

$$\tilde{\sigma}_m(x) = \inf\{\|x - \sum_{k\in A} e_k^*(x) e_k\| : \ |A| \le m\}.$$

Note that

$$\sigma_m(x) \le \tilde{\sigma}_m(x) \le \|R_m(x)\| \to 0 \qquad \text{as } m \to \infty.$$

Let us say that a basis (e_n) is *almost greedy* if there is a constant C so that:

$$\|H_m x\| \le C\tilde{\sigma}_m(x). \tag{2.3}$$

We will say that a basis (e_n) is *partially greedy* if there is a constant C so that for any $x \in X,\ m \in \mathbb{N}$,

$$\|H_m(x)\| \le C\|R_m x\|. \tag{2.4}$$

It is clear that for any basis we have the following implications:

greedy ⇒ almost greedy ⇒ partially greedy ⇒ quasi-greedy.

Next we prove two useful lemmas concerning quasi-greedy bases. These are both essentially due to Wojtaszczyk [9]. The first lemma says that every quasi-greedy basis is *unconditional for constant coefficients.*

Lemma 2.1. *Suppose* $(e_n)_{n\in\mathbb{N}}$ *has quasi-greedy constant* K. *Suppose* A *is a finite subset of* $\mathbb{N}$. *Then, for every choice of signs* $\epsilon_j = \pm 1$, *we have*

$$\frac{1}{2K}\|\sum_{j\in A} e_j\| \le \|\sum_{j\in A}\epsilon_j e_j\| \le 2K\|\sum_{j\in A} e_j\|, \tag{2.5}$$

and hence for any real numbers $(a_j)_{j\in A}$

$$\|\sum_{j\in A} a_j e_j\| \le 2K\max_{j\in A}|a_j|\|\sum_{j\in A} e_j\|. \tag{2.6}$$

Proof. First note that if $B \subset A$ and $\varepsilon > 0$ then

$$\|\sum_{j\in B}(1+\varepsilon)e_j\| \le K\|\sum_{j\in B}(1+\varepsilon)e_j + \sum_{j\in A\setminus B} e_j\|.$$

Letting $\varepsilon \to 0$, we obtain $\|\sum_{j\in B} e_j\| \le K\|\sum_{j\in A} e_j\|$, and hence for any choice of signs $\epsilon_j = \pm 1$, we have

$$\|\sum_{j\in A}\epsilon_j e_j\| \le 2K\|\sum_{j\in A} e_j\|.$$

This gives the right-hand inequality in (2.5) and the left-hand inequality is similar. By convexity (2.6) follows immediately. □

Lemma 2.2. *Suppose* $(e_n)_{n\in\mathbb{N}}$ *has quasi-greedy constant* K. *Suppose* $x \in X$ *has greedy ordering* ρ. *Then*

$$|e^*_{\rho(m)}(x)|\|\sum_{j=1}^{m} e_{\rho(m)}\| \le 4K^2\|x\| \tag{2.7}$$

and hence if A *is any subset of* $\mathbb{N}$ *and* $(a_j)_{j\in A}$ *any real numbers,*

$$\min_{j\in A}|a_j|\|\sum_{j\in A} e_j\| \le 4K^2\|\sum_{j\in A} a_j e_j\| \tag{2.8}$$

Proof. We prove (2.7) and then (2.8) is immediate. Let $a_j = e^*_j(x)$. Let $\epsilon_j = \operatorname{sgn} a_j$ and put $1/|a_0| = 0$. Then

$$\begin{aligned}|a_{\rho(m)}|\|\sum_{j=1}^{m}\epsilon_{\rho(j)}e_{\rho(j)}\| &= |a_{\rho(m)}|\|\sum_{j=1}^{m}\left(\frac{1}{|a_{\rho(j)}|} - \frac{1}{|a_{\rho(j-1)}|}\right)(H_{j-1}(x) - H_m(x))\| \\ &\le 2K\|x\|.\end{aligned}$$

We then use (2.5). □

We conclude this section by considering direct and inverse theorems for approximation with regard to almost greedy bases. For a basis (e_n) and greedy ordering ρ, denote, for $x \in X$,

$$a_k(x) = |e^*_{\rho(k)}(x)|.$$

The following theorem was proved in [8].

Theorem 2.3. *Let $1 < p < \infty$ and let (e_n) be a greedy basis with $\phi(n) \asymp n^{1/p}$. Then for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence:*

$$\sum_n \sigma_n(x)^q n^{rq-1} < \infty \Leftrightarrow \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

We generalize this theorem as follows.

Theorem 2.4. *Let $1 < p < \infty$ and let (e_n) be a democratic quasi-greedy basis with $\phi(n) \asymp n^{1/p}$. Then for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence:*

$$\sum_n \|H_n(x)\|^q n^{rq-1} < \infty \Leftrightarrow \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

The proof of this theorem is similar to the proof of Theorem 2.3 and is based on the following lemmas which are analogous to the corresponding lemmas from [8].

Lemma 2.5. *Let (e_n) be a democratic quasi-greedy basis with $\phi(n) \asymp n^{1/p}$. Then there exists a constant C such that for any two positive integers $N < M$ and any $x \in X$, we have*

$$a_M(x) \le C\|H_N(x)\|(M-N)^{-1/p}.$$

Proof. This lemma follows from (2.8) of Lemma 2.2. □

Lemma 2.6. *Let (e_n) be a democratic quasi-greedy basis with $\phi(n) \asymp n^{1/p}$. Then there exists a constant C such that for any sequence $m_0 < m_1 < \dots$ of non-negative integers, we have*

$$\|H_{m_s}(x)\| \le C \sum_{l=s}^{\infty} a_{m_l}(x)(m_{l+1} - m_l)^{1/p}.$$

Proof. This lemma follows from (2.6) of Lemma 2.1. □

By Theorem 3.3 below we get that a democratic quasi-greedy basis is almost greedy and also has the following property (setting $\lambda = 2$ in (3) of Theorem 3.3):

$$\sigma_{2n}(x) \le \|H_{2n}(x)\| \le C\sigma_n(x)$$

This inequality implies that

$$\sum_n \|H_n(x)\|^q n^{rq-1} < \infty \Leftrightarrow \sum_n \sigma_n(x)^q n^{rq-1} < \infty.$$

Therefore Theorem 2.3 holds with the assumption that (e_n) is greedy replaced by the assumption that (e_n) is almost greedy, which yields Theorem 2.4.

3. Democratic and conservative bases

We recall that a basis (e_n) in a Banach space X is called *democratic* if there is a constant Δ such that

$$\|\sum_{k\in A} e_k\| \le \Delta \|\sum_{k\in B} e_k\| \quad \text{if } |A| \le |B|. \tag{3.1}$$

This concept was introduced in [5]. The following characterization of greedy bases was also proved in [5].

Theorem 3.1. *A basis (e_n) is greedy if and only if it is unconditional and democratic.*

For a basis (e_n) we define the *fundamental function* $\varphi(n)$ by

$$\varphi(n) = \sup_{|A|\le n} \|\sum_{k\in A} e_k\|.$$

The *dual fundamental function* is given by

$$\varphi^*(n) = \sup_{|A|\le n} \|\sum_{k\in A} e_k^*\|.$$

Note that φ (and φ^*) is subadditive (i.e. $\varphi(m+n) \le \varphi(m)+\varphi(n)$) and increasing. It may also be seen that $\varphi(n)/n$ (and $\varphi^*(n)/n$) is decreasing since for any set A with $|A| = n$ we have

$$\sum_{k\in A} e_k = \frac{1}{n-1} \sum_{k\in A} \sum_{j\ne k} e_j.$$

It follows that for any set A and any scalars $(a_j : \ j \in A)$ we have:

$$\|\sum_{j\in A} a_j e_j\| \le 2\varphi(|A|) \max_{j\in A} |a_j|. \tag{3.2}$$

It is clear that (e_k) is democratic with constant Δ in (3.1) if and only if

$$\Delta^{-1}\varphi(|A|) \le \|\sum_{k\in A} e_k\| \le \varphi(|A|), \qquad |A| < \infty. \tag{3.3}$$

Lemma 3.2. *Let (e_n) be a democratic quasi-greedy basis. Let K be the quasi-greedy constant and Δ the democratic constant. Then for $x \in X$ if ρ is the quasi-greedy ordering,*

$$|e^*_{\rho(m)}(x)| \le \frac{4K^2\Delta}{\varphi(m)}\|x\|, \tag{3.4}$$

and

$$\sup_{k\in\mathbb{N}} |e^*_k(H_m x)| \le \frac{4K^2\Delta}{\varphi(m+1)}\|x\|. \tag{3.5}$$

Proof. This follows directly from (3.3) and Lemma 2.2 (2.7). □

Next we compare almost greedy bases with greedy bases. Essentially in an almost greedy basis the convergence of the TGA is almost optimal. It follows from (2) below and [9] that any conditional quasi-greedy basis of a Hilbert space is actually almost greedy. See also [3] for a conditional almost greedy basis of ℓ_1.

Theorem 3.3. *Suppose (e_n) is a basis of a Banach space. The following are equivalent:*
(1) (e_n) is almost greedy.
(2) (e_n) is quasi-greedy and democratic.
(3) For any (respectively, every) $\lambda > 1$ there is a constant $C = C_\lambda$ such that

$$\|H_{[\lambda m]}x\| \le C_\lambda \sigma_m(x).$$

Proof. We start by showing (1) implies (2). It is immediate that (e_n) is quasi-greedy. Now suppose $|A| \le |B|$. Suppose $\delta > 0$ and define

$$x = \sum_{j\in A} e_j + \sum_{j\in B\setminus A} (1+\delta)e_j.$$

Then if $r = |B \setminus A|$ we have $H_r(x) = \sum_{j\in A} e_j$. However

$$\tilde{\sigma}_r(x) \le \|\sum_{j\in B} e^*_j(x)e_j\| \le \|\sum_{j\in B} e_j\| + \delta\|\sum_{j\in B\setminus A} e_j\|.$$

Letting $\delta \to 0$, it follows from (2.3) that (e_n) is democratic.

Next we show that (2) implies (1) so that (1) and (2) are equivalent. Suppose $x \in X$ and $m \in \mathbb{N}$. Let

$$G_m(x) = \sum_{j\in A} e^*_j(x)e_j$$

where $|A| = m$. Suppose $|B| = r \le m$. Then

$$H_m(x) = (x - \sum_{j\in B} e^*_j(x)e_j) + \sum_{j\in B\setminus A} e^*_j(x)e_j - \sum_{j\in A\setminus B} e^*_j(x)e_j.$$

Then $|B \setminus A| \le s := |A \setminus B|$. Thus

$$\|\sum_{j\in B\setminus A} e_j^*(x)e_j\| \le 2K(\max_{j\in B\setminus A}|e_j^*(x)|)\varphi(s)$$

(by (2.6))

$$\le 2K(\min_{j\in A\setminus B}|e_j^*(x)|)\varphi(s)$$
$$\le 8K^3\Delta\|\sum_{j\in A\setminus B} e_j^*(x)e_j\|$$

(by (3.4))

$$= 8K^3\Delta\|G_s(x-\sum_{j\in B} e_j^*(x)e_j)\|$$
$$\le 8K^4\Delta\|(x-\sum_{j\in B} e_j^*(x)e_j)\|.$$

We also have:

$$\|\sum_{j\in A\setminus B} e_j^*(x)e_j\| = \|G_s(x-\sum_{j\in B} e_j^*(x)e_j)\|.$$

Thus it follows that

$$\|H_m(x)\| \le (8K^4\Delta + K + 1)\|x - \sum_{j\in B} e_j^*(x)e_j\|$$

and so, optimizing over B with $|B| \le m$,

$$\|H_m(x)\| \le (8K^4\Delta + K + 1)\tilde{\sigma}_m(x).$$

Let us prove that (2) implies (3) for every $\lambda > 1$. Assume K is the quasi-greedy constant and Δ is the democratic constant. Assume $m, r \in \mathbb{N}$. For $x \in X$ and A a finite subset of cardinality m, let $v = \sum_{j\notin A} e_j^*(x)e_j$. Now suppose y is such that $e_j^*(y) \ne e_j^*(x)$ only if $j \in A$. Then

$$G_r(y) = \sum_{j\in B} e_j^*(y)e_j$$

where $|B| = r$. Let $|A\cap B| = s$ where $0 \le s \le \min(r,m)$. Then

$$H_r(y) - H_{r-s}(v) = H_s(y-v) = \sum_{j\in A\setminus B} e_j^*(y)e_j.$$

Now by (3.5)

$$\max_{j\in A\setminus B}|e_j^*(y)| \le \frac{4K^2\Delta}{\varphi(r+1)}\|y\|.$$

Hence by (2.6)

$$\|H_r(y) - H_{r-s}(v)\| \leq \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|. \tag{3.6}$$

For $\epsilon > 0$ we can choose y so that $\|y\| < \sigma_m(x) + \epsilon$ and $\{j : e_j^*(y) \neq e_j^*(x)\}$ is contained in a set A of cardinality m as above. Note that

$$\tilde{\sigma}_{m+r}(x) \leq \tilde{\sigma}_{m+r-s}(x) \leq \|H_{r-s}(v)\|$$

and hence (3.6) and the triangle inequality yield

$$\begin{aligned}\tilde{\sigma}_{m+r}(x) &\leq \|H_{r-s}(v)\| \\ &\leq \|H_r(y)\| + \|H_r(y) - H_{r-s}(v)\| \\ &\leq K\|y\| + \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|.\end{aligned}$$

Since $\|y\| \leq \sigma_m(x) + \epsilon$ and ϵ is arbitrary, we obtain

$$\tilde{\sigma}_{m+r}(x) \leq (\frac{8K^3\Delta\varphi(m)}{\varphi(r+1)} + K)\sigma_m(x).$$

Next suppose $\lambda > 1$ and $r = [\lambda m] - m$. Now $\varphi(m)/\varphi(r+1) \leq m/(r+1)$, so we have

$$\tilde{\sigma}_{[\lambda m]}(x) \leq (\frac{8K^3\Delta}{\lambda - 1} + K)\sigma_m(x)$$

This implies (3) with $C_\lambda \asymp (\lambda - 1)^{-1}$.

It remains to show (3) (for some fixed $\lambda > 1$) implies (2). That (e_n) is quasi-greedy is immediate. Note that if $|D| = [\lambda m]$, then

$$\|\sum_{j\in D} e_j\| \leq \varphi(\lambda m) \leq \lambda\varphi(m).$$

So to prove that (e_n) is democratic it is enough to show that

$$\|\sum_{j\in D} e_j\| \geq \varphi(m)/C_\lambda.$$

Suppose $|A| \leq m < \infty$. For any set B of cardinality $[\lambda m]$ disjoint from A we have (by a statement argument as in the case (1) implies (2))

$$\|\sum_{j\in A} e_j\| \leq C_\lambda \sigma_m(\sum_{j\in A\cup B} e_j) \leq C_\lambda \|\sum_{j\in D} e_j\|$$

whenever $D \subset A \cup B$ with $|D| \geq [\lambda m]$. Thus, maximizing over all A with $|A| \leq m$,

$$\inf_{|D|=[\lambda m]} \|\sum_{j\in D} e_j\| \geq \varphi(m)/C_\lambda$$

and so (e_j) is democratic. $\square$

If A, B are subsets of $\mathbb{N}$ we use the notation $A < B$ to mean that $m \in A, n \in B$ implies $m < n$. We write $n < A$ for $\{n\} < A$. Let us define a basis (e_n) to be *conservative* if there is a constant Γ such that

$$\|\sum_{k\in A} e_k\| \le \Gamma \|\sum_{k\in B} e_k\| \qquad \text{if } |A| \le |B| \text{ and } A < B. \tag{3.7}$$

The analogue of Theorem 3.1 and Theorem 3.3 is:

Theorem 3.4. *A basis (e_n) is partially greedy if and only if it is quasi-greedy and conservative.*

Proof. Clearly a partially greedy basis is also quasi-greedy. Suppose (e_n) is partially greedy (with constant C in (2.4)) and $A < B$ with $|A| = |B| = m$. Let $r = \max A$. Let $D = [1, r] \setminus A$ and then for $\delta > 0$ let

$$x = \sum_{k\in A} e_k + (1+\delta) \sum_{k\in D\cup B} e_k.$$

Then

$$\|H_r(x)\| = \|\sum_{k\in A} e_k\|$$

and

$$\|R_r(x)\| = (1+\delta)\|\sum_{k\in B} e_k\|$$

so that letting $\delta \to 0$ gives (3.7) with $\Gamma = C$. $\square$

Conversely, let us suppose (e_n) is quasi-greedy with constant K and conservative with constant Γ. Suppose $x \in X$ and $m \in \mathbb{N}$. Let ρ be the greedy ordering for x. Then let $D = \{\rho(j) : \ j \le m, \ \rho(j) \le m\}$, and $B = \{\rho(j) : \ j \le m, \ \rho(j) > m\}$. Let $A = [1, m] \setminus D$. Then $|A| = |B| = r$, say, and $A < B$. Now

$$\|\sum_{k\in B} e_k^*(x) e_k\| = \|G_r(R_m x)\| \le K\|R_m x\|.$$

Also

$$\begin{aligned}\|\sum_{k\in A} e_k^*(x) e_k\| &\le 2K(\max_{k\in A} |e_k^*(x)|)\|\sum_{k\in A} e_k\| \\ &\le 2K\Gamma(\min_{k\in B} |e_k^*(x)|)\|\sum_{k\in B} e_k\| \\ &\le 8K^3\Gamma\|\sum_{k\in B} e_k^*(x) e_k\|\end{aligned}$$

(by (2.8))

$$\leq 8K^4\Gamma\|R_m x\|.$$

Combining gives us

$$\begin{aligned}\|H_m x\| &\leq \|R_m x\| + \|\sum_{k\in A} e_k^*(x)e_k\| + \|\sum_{k\in B} e_k^*(x)e_k\| \\ &\leq (8K^4\Gamma + K + 1)\|R_m x\|.\end{aligned}$$

□

4. Bidemocratic bases

Suppose (e_n) is a democratic basis. We shall say that (e_n) has the *upper regularity property (URP)* if there exists an integer $r > 2$ so that

$$\varphi(rn) \leq \tfrac{1}{2} r\varphi(n) \qquad n \in \mathbb{N} \tag{4.1}$$

This of course implies $\varphi(r^k n) \leq 2^{-k} r^k \varphi(n)$ and is therefore easily equivalent to the existence of $0 < \beta < 1$ and a constant C so that if $m > n$,

$$\varphi(m) \leq C\left(\frac{m}{n}\right)^{\beta} \varphi(n). \tag{4.2}$$

We say (e_n) has the *lower regularity property (LRP)* if there exists $r > 1$ so that for all $n \in \mathbb{N}$ we have

$$\varphi(rn) \geq 2\varphi(n) \qquad n \in \mathbb{N}. \tag{4.3}$$

This is similarly equivalent to the existence of $0 < \alpha < 1$ and $c > 0$ so that if $m > n$

$$\varphi(m) \geq c\left(\frac{m}{n}\right)^{\alpha} \varphi(n). \tag{4.4}$$

Let us recall that a Banach space X has (Rademacher) type $1 < p \leq 2$ if there is a constant C so that

$$(\operatorname*{Ave}_{\epsilon_j=\pm 1} \|\sum_{j=1}^{n} \epsilon_j x_j\|^p)^{\frac{1}{p}} \leq C(\sum_{j=1}^{n} \|x_j\|^p)^{\frac{1}{p}} \qquad x_1, \cdots, x_n \in X, n \in \mathbb{N}.$$

The least such constant C is called the type p-constant $T_p(X)$. X has (Rademacher) cotype $2 \leq q < \infty$ if there exists a constant C such that

$$(\sum_{j=1}^{n} \|x_j\|^q)^{\frac{1}{q}} \leq C(\operatorname*{Ave}_{\epsilon_j=\pm 1} \|\sum_{j=1}^{n} \epsilon_j x_j\|^q)^{\frac{1}{q}} \qquad x_1, \cdots, x_n \in X, n \in \mathbb{N}.$$

The least such constant C is called the cotype q-constant $C_q(X)$.

Proposition 4.1. *(1) If (e_n) is an almost greedy basis of a Banach space with non-trivial cotype then (e_n) has (LRP).*
(2) If (e_n) is an almost greedy basis of a Banach space with non-trivial type then (e_n) has (LRP) and (URP).

Proof. (1) Suppose K is the quasi-greedy constant of (e_n) and Δ is the democratic constant. Suppose X has cotype $q < \infty$ with constant $C_q(X)$. Let $B_1, \cdots, B_m$ be disjoint sets with $|B_k| = n$ and let $A = \cup_{k=1}^m B_k$. Then, using Lemma 2.1, (2.5), and (3.3)

$$\begin{aligned} m^{\frac{1}{q}}\varphi(n) &\le \Delta \left(\sum_{k=1}^m \| \sum_{j\in B_k} e_j \|^q \right)^{\frac{1}{q}} \\ &\le 2K\Delta \left(\sum_{k=1}^m \operatorname*{Ave}_{\epsilon_j=\pm 1} \| \sum_{j\in B_k} \epsilon_j e_j \|^q \right)^{\frac{1}{q}} \\ &\le 2K\Delta C_q(X) \left(\operatorname*{Ave}_{\epsilon_j=\pm 1} \| \sum_{j\in A} \epsilon_j e_j \|^q \right)^{\frac{1}{q}} \\ &\le 4K\Delta C_q(X)\varphi(mn). \end{aligned}$$

It is clear this implies (4.4) for some suitable constant $c > 0$ and $\alpha = \frac{1}{q}$.

(2) Since nontrivial type implies nontrivial cotype we obtain (LRP) immediately. The proof of (URP) (with $\beta = \frac{1}{p}$ when X has type p) is very similar. Using the same notation and assuming X has type $p > 1$ with constant $T_p(X)$ we have:

$$\begin{aligned} \varphi(mn) &\le 2K\Delta (\operatorname*{Ave}_{\epsilon_j=\pm 1} \| \sum_{j\in A} \epsilon_j e_j \|^p)^{\frac{1}{p}} \\ &\le 2K\Delta T_p(X) \left(\sum_{k=1}^m \operatorname*{Ave}_{\epsilon_j=\pm 1} \| \sum_{j\in B_k} \epsilon_j e_j \|^p \right)^{\frac{1}{p}} \\ &\le 4K\Delta T_p(X) m^{\frac{1}{p}} \varphi(n). \end{aligned}$$

This implies (4.2) for suitable constants. □

We now say that a basis (e_n) is *bidemocratic* if there is a constant Δ so that

$$\varphi(n)\varphi^*(n) \le \Delta n. \tag{4.5}$$

Proposition 4.2. *If (e_n) is bidemocratic (with constant Δ) then (e_n) and (e_n^*) are both democratic (with constant Δ) and are both unconditional for constant coefficients.*

Proof. If A is any finite set we have

$$|A| \leq \|\sum_{j\in A} e_j^*\|\|\sum_{j\in A} e_j\| \leq \varphi^*(|A|)\|\sum_{j\in A} e_j\|.$$

Hence

$$\Delta^{-1}\varphi(|A|) \leq \|\sum_{j\in A} e_j\|$$

and so (e_n) is democratic with constant Δ. Let $(\epsilon_j)_{j\in A}$ be any choice of signs ± 1. Then

$$|A| \leq \|\sum_{j\in A} \epsilon_j e_j^*\|\|\sum_{j\in A} \epsilon_j e_j\| \leq 2\varphi^*(|A|)\|\sum_{j\in A} \epsilon_j e_j\|.$$

Hence

$$\frac{1}{2\Delta}\varphi(|A|) \leq \|\sum_{j\in A} \epsilon_j e_j\| \leq 2\varphi(|A|).$$

Hence (e_n) is unconditional for constant coefficients. Similar calculations work for (e_j^*) to obtain the theorem. □

Proposition 4.3. *A basis (e_n) is bidemocratic if and only if there is a constant C so that for any finite set $A \subset \mathbb{N}$,*

$$\|\sum_{k\in A} e_k\|\|\sum_{k\in A} e_k^*\| \leq C|A|. \tag{4.6}$$

Proof. One direction is trivial. Assume (4.6) holds with $C \geq 1$. Suppose $n \in \mathbb{N}$. By passing to an equivalent norm on X, if necessary, we may assume that (e_n) and (e_n^*) are both monotone. There exist $A, B \subset \mathbb{N}$ with $|A| \leq n, |B| \leq n$ and

$$\|\sum_{j\in A} e_j\| \geq \tfrac{1}{2}\varphi(n), \qquad \|\sum_{j\in B} e_j^*\| \geq \tfrac{1}{2}\varphi^*(n).$$

By monotonicity of (e_n) and (e_n^*) we may assume that $|A| = |B| = n$. Let $D = A \cup B$, $E = D \setminus A$.

If $\|\sum_{j\in D} e_j\| \geq \frac{1}{8C}\varphi(n)$ and $\|\sum_{j\in D} e_j^*\| \geq \frac{1}{8C}\varphi^*(n)$ we obtain immediately that

$$\varphi(n)\varphi^*(n) \leq 2^6 C^3 |D| \leq 2^7 C^3 n.$$

Consider when one of these inequalities fails; we need only treat the case $\|\sum_{j\in D} e_j\| < \frac{1}{8C}\varphi(n)$. Then

$$\|\sum_{j\in E} e_j\| \geq \|\sum_{j\in A} e_j\| - \|\sum_{j\in D} e_j\| > \frac{\varphi(n)}{2} - \frac{\phi(n)}{8C} > \frac{\varphi(n)}{4}$$

and thus, as $|E| \le n$, (4.6) gives

$$\|\sum_{j\in E} e_j^*\| \le 4Cn\varphi(n)^{-1}.$$

We also have from (4.6) that

$$\|\sum_{j\in A} e_j^*\| \le 2Cn\varphi(n)^{-1}.$$

Hence

$$\|\sum_{j\in D} e_j^*\| \le 6Cn\varphi(n)^{-1}$$

and so

$$n \le |D| \le \|\sum_{j\in D} e_j\|\|\sum_{j\in D} e_j^*\| \le (\frac{6Cn}{\varphi(n)})(\frac{\varphi(n)}{8C}) = \frac{3n}{4}$$

which is a contradiction. □

Proposition 4.4. *If (e_n) is a democratic quasi-greedy basis with (URP) then (e_n) is bidemocratic.*

Proof. We assume (4.2) holds, that (e_n) is quasi-greedy with constant K and democratic with constant Δ. Suppose A is a finite subset of $\mathbb{N}$. Pick $x \in X$ so that $\|x\| = 1$ and $\sum_{j\in A} e_j^*(x) > \frac{1}{2}\|\sum_{j\in A} e_j^*\|$. Let ρ be the greedy ordering for x. Then by (3.5), if $|A| = n$,

$$\begin{aligned}
\varphi(n)\|\sum_{j\in A} e_j^*\| &\le 2\varphi(n)\sum_{j\in A}|e_j^*(x)| \\
&\le 2\varphi(n)\sum_{k=1}^{n}|e_{\rho(k)}^*(x)| \\
&\le 8K^2\Delta\sum_{k=1}^{n}\frac{\varphi(n)}{\varphi(k)} \\
&\le 8K^2\Delta C n^{\beta}\sum_{k=1}^{n}k^{-\beta} \\
&\le C_1 n
\end{aligned}$$

for a suitable constant C_1. This implies $\varphi(n)\varphi^*(n) \le C_1 n$. □

Corollary 4.5. *Let (e_n) be a quasi-greedy basis for a Hilbert space. Then (e_n) is bidemocratic.*

Proof. Wojtaszczyk [9] proved that (e_n) is democratic and that $\varphi(n) \asymp \sqrt{n}$. So the result follows from Proposition 4.4. □

Remark 4.6. Proposition 4.4 breaks down for bases that are not quasi-greedy. To see this, let (e_n^p) be the unit vector basis of ℓ_p. We define a normalized basis (f_n) of $\ell_2 \oplus_2 \ell_p$ as follows:

$$f_{2n-1} = \frac{1}{\sqrt{2}}(e_n^2 + e_n^p); \qquad f_{2n} = \frac{1}{2}e_n^2 + \frac{\sqrt{3}}{2}e_n^p.$$

Suppose that $1 < p < 2$. It is easy to check that (f_n) and (f_n^*) are both democratic and unconditional for constant coefficients, that $\varphi(n) \asymp n^{1/p}$, and that $\varphi^*(n) \asymp \sqrt{n}$. So both (f_n) and (f_n^*) have (URP) but (f_n) is not bidemocratic.

5. Duality of almost greedy bases

Theorem 5.1. *Let (e_n) be a greedy basis with (URP). Then (e_n^*) is a greedy basic sequence. In particular, if (e_n) is a greedy basis of a Banach space X with non-trivial type then (e_n^*) is a greedy basis of X^*.*

Proof. Since (e_n^*) is automatically unconditional this follows from Proposition 4.4 and Theorem 3.1. The second part follows from Proposition 4.1; note that any space with nontrivial type and an unconditional basis is reflexive by James's theorem [4]. □

Remark 5.2. In [3] there is an example of an almost greedy basis (e_n) of ℓ_1 such that (e_n^*) is not unconditional for constant coefficients, thus not quasi-greedy. The example localizes to give a quasi-greedy basis of the reflexive space $(\sum \oplus \ell_1^n)_2$ whose dual basis is not quasi-greedy. On the other hand, it follows from Corollary 4.5 above and Theorem 5.4 below that in a Hilbert space the dual basis of a quasi-greedy basis is always quasi-greedy (in fact, both the basis and its dual are almost greedy).

Corollary 5.3. *If $1 < p < \infty$ the space L_p has a greedy basis not equivalent to a rearranged subsequence of the Haar system.*

Proof. For $p > 2$ Wojtaszczyk [9] constructed such a basis with $\varphi(n) \asymp n^{1/p}$, hence with (URP). The case $p < 2$ follows by duality using Theorem 5.1. □

Theorem 5.4. *Let (e_n) be a quasi-greedy basis of a Banach space X. Then the following are equivalent:*
(1) (e_n) is bidemocratic.
(2) (e_n) and (e_n^*) are both almost greedy.
(3) (e_n) and (e_n^*) are both partially greedy.

Proof. We first prove (1) implies (2). Let Δ denote the bidemocratic constant. In fact by Proposition 4.2 we only need show that (e_n^*) is quasi-greedy. Let us denote by G_m^* and H_m^* the greedy operator and greedy remainder operators associated to the dual basic sequence (e_n^*). Suppose $x^* \in X^*$ and $x \in X$.

First note that if $|A| = m$ then

$$\sum_{j\in A} |x^*(e_j)| \le \|x^*\| \sup_{\epsilon_j=\pm 1} \|\sum_{j\in A} \epsilon_j e_j\| \\ \le 2\varphi(m)\|x^*\|.$$

Hence

$$\sup_{j\in\mathbb{N}} |(H_m^* x^*)(e_j)| \le 2\frac{\varphi(m+1)}{m+1}\|x^*\|. \tag{5.1}$$

On the other hand (3.5) implies that

$$\sup_{j\in\mathbb{N}} |e_j^*(H_m(x))\| \le \frac{4K^2\Delta}{\varphi(m+1)}\|x\|. \tag{5.2}$$

Suppose $G_m(x) = \sum_{j\in A} e_j^*(x)e_j$ and $G_m^*(x^*) = \sum_{j\in B} x^*(e_j)e_j^*$ where $|A| = |B| = m$. Then

$$|(H_m^* x^*)(G_m(x))| = |(\sum_{j\in A\setminus B} x^*(e_j)e_j^*)(x)| \\ \le \|\sum_{j\in A\setminus B} x^*(e_j)e_j^*\|\|x\| \\ \le 4\frac{\varphi(m+1)\varphi^*(m)}{m+1}\|x\|\|x^*\|$$

(by (3.2) and (5.1))

$$\le 4\Delta\|x\|\|x^*\|.$$

Also,

$$|(G_m^* x^*)(H_m(x))| = |x^*(\sum_{j\in B\setminus A} e_j^*(x)e_j)| \\ \le \|x^*\|\frac{4K^2\Delta\|x\|}{\varphi(m+1)}(2\varphi(m))$$

(by (5.2))

$$\le 8K^2\Delta\|x\|\|x^*\|.$$

Now

$$G_m^* x^*(x) = x^*(G_m x) - (H_m^* x^*)(G_m x) + G_m^*(x^*)(H_m x).$$

Hence

$$|G_m^* x^*(x)| \le (K + 4\Delta + 8K^2\Delta)\|x\|\|x^*\|$$

so that

$$\|G_m^* x^*\| \le (K + 4\Delta + 8K^2\Delta)\|x^*\|.$$

This implies (e_n^*) is a quasi-greedy basic sequence, and proves (1) implies (2).

Of course (2) implies (3) so it remains to prove (3) implies (1). Let us assume that K is a quasi-greedy constant for both (e_n) and (e_n^*), and that Γ is a conservative constant for both (e_n) and (e_n^*).

Suppose A is any finite subset of $\mathbb{N}$. For $x \in [e_j]_{j \notin A}$, let $y = \sum_{j\in A} e_j + x$. First suppose that $|e_j^*(x)| \ne 1$ for all j. Then

$$\begin{aligned}\|\sum_{j\in A} e_j\| &\le \|\sum_{|e_j^*(y)|\le 1} e_j^*(y)e_j\| + \|\sum_{|e_j^*(y)|<1} e_j^*(y)e_j\| \\ &\le 2K\|y\|.\end{aligned}$$

By continuity, $\|\sum_{j\in A} e_j\| \le 2K\|y\|$ for all $x \in [e_j]_{j\notin A}$. Thus, by Nikol'skii's Duality Theorem (see e.g. [6]), there exists $x^* \in [e_j]_{j\in A}$ with $\|x^*\| = 1$ and

$$|x^*(\sum_{j\in A} e_j)| \ge \frac{1}{2K}\|\sum_{j\in A} e_j\|. \tag{5.3}$$

Now suppose $m \in \mathbb{N}$. Choose A_0, B_0 with $|A_0|, |B_0| \le m$ and

$$\|\sum_{j\in A_0} e_j\| \ge \tfrac{1}{2}\varphi(m), \qquad \|\sum_{j\in B_0} e_j^*\| \ge \tfrac{1}{2}\varphi^*(m).$$

Now let A be any subset of $\mathbb{N}$ with $|A| = 2m$ and $A > \max(A_0, B_0)$.

Note that if $D \subset A$ and $|D| \ge m$, then since (e_n) and (e_n^*) are conservative with constant Γ,

$$\|\sum_{j\in D} e_j\| \ge \frac{1}{2\Gamma}\varphi(m), \qquad \|\sum_{j\in D} e_j^*\| \ge \frac{1}{2\Gamma}\varphi^*(m). \tag{5.4}$$

Let us choose $u^* \in [e_j^*]_{j\in A}$ such that $\sum_{j\in A} |u_j^*(e_j)|^2$ is minimized subject to $\|u^*\| \le 1$ and

$$\sum_{j\in A} u^*(e_j) \ge \frac{\varphi(m)}{4\Gamma K}. \tag{5.5}$$

This is possible by (5.3) and (5.4).

Now let $G^*_m(u^*) = \sum_{j\in B} u^*(e_j)e_j^*$ where $B \subset A$ and $|B| = m$. Let $D = A \setminus B$. We observe that by (2.7) we have

$$\min_{j\in B} |u^*(e_j)| \|\sum_{j\in B} e_j^*\| \le 4K^2$$

and hence by (5.4)

$$\min_{j\in B} |u^*(e_j)| \le \frac{8K^2\Gamma}{\varphi^*(m)}. \tag{5.6}$$

We then again use (5.3) to find $v^* \in [e_j^*]_{j\in D}$ with $\|v^*\| = 1$ and

$$\sum_{j\in D} v^*(e_j) \ge \frac{\varphi(m)}{4\Gamma K}.$$

It follows from the minimality assumption on u^* that

$$\sum_{j\in A}((1-t)u^*(e_j) + tv^*(e_j))^2 \ge \sum_{j\in A}(u^*(e_j))^2$$

for $0 \le t \le 1$ and so using (2.8) and (5.6),

$$\begin{aligned}
\sum_{j\in A} u^*(e_j)^2 &\le \sum_{j\in A} u^*(e_j)v^*(e_j) \\
&\le \min_{j\in B} |u^*(e_j)| \sum_{j\in D} |v^*(e_j)| \\
&\le \frac{8K^2\Gamma}{\varphi^*(m)} \max_{\epsilon_j=\pm 1} \|\sum_{j\in D} \epsilon_j e_j\| \\
&\le \frac{16K^2\Gamma\varphi(m)}{\varphi^*(m)}.
\end{aligned}$$

Thus from (5.5)

$$\begin{aligned}
(\varphi(m))^2 &\le 2^4\Gamma^2K^2 \left(\sum_{j\in A} |u^*(e_j)|\right)^2 \\
&\le 2^4\Gamma^2K^2 m \sum_{j\in A} u^*(e_j)^2 \\
&\le \frac{2^8\Gamma^3K^4 m\varphi(m)}{\varphi^*(m)}
\end{aligned}$$

which gives the estimate

$$\varphi(m)\varphi^*(m) \le 2^8\Gamma^3K^4 m,$$

so that (e_n) is bidemocratic. □

Corollary 5.5. *Let X be a Banach space with non-trivial type. If (e_n) is an almost greedy basis of X then (e_n^*) is an almost greedy basic sequence in X^*.*

Proof. This follows directly from Theorem 5.4 and Proposition 4.4. □

Corollary 5.6. *Suppose that (e_n) and (e_n^*) are both partially greedy and that $\varphi(n) \asymp n$. Then (e_n) is equivalent to the unit vector basis of ℓ_1.*

Proof. By Theorem 5.4, (e_n) is bidemocratic. Hence

$$\varphi^*(n) \asymp n/\varphi(n) \asymp 1.$$

But this implies that (e_n^*) is equivalent to the unit vector basis of c_0, which gives the result. □

Example 5.7. Let us conclude this section by showing that if $\varphi : \mathbb{N} \to (0, \infty)$ is an increasing function satisfying $\varphi(1) = 1$ and $\varphi(n)/n$ is decreasing, but failing (4.1) then it is possible to construct a Banach space with a greedy basis (e_n) with a fundamental function equivalent to $\varphi(n)$ and such that the dual basic sequence (e_n^*) is not greedy. This will show that the preceding theorem is, in some sense, sharp. In Example 5.9, we will show under very mild additional conditions on φ how to make a reflexive example.

Let us define the sequence space X_φ to be the completion of c_{00} for the norm

$$\|\xi\|_\varphi = \sup_n \sup_{\substack{|A|=n \\ n<A}} \frac{\varphi(n)}{n} \sum_{k \in A} |\xi_k|.$$

It is clear that the canonical basis is unconditional. It also democratic since if $|A| = n$,

$$\tfrac{1}{2}\varphi(n) \le \|\sum_{k \in A} e_k\| \le \varphi(n) \tag{5.7}$$

Let us suppose the dual basic sequence (e_n^*) is democratic with democratic constant Δ. We note that if $A > n$ then

$$\|\sum_{k \in A} e_k^*\| \le n/\varphi(n).$$

It follows from the democratic assumption that

$$\|\sum_{k=1}^n e_k^*\| \le \Delta n/\varphi(n).$$

Now consider

$$\xi = \sum_{k=1}^{n} \frac{1}{\varphi(k)} e_k.$$

Clearly $\|\xi\| \leq 1$ and so

$$\|\sum_{k=1}^{n} e_k^*\| \geq \sum_{k=1}^{n} \frac{1}{\varphi(k)}.$$

We deduce that

$$\sum_{k=1}^{n} \frac{1}{\varphi(k)} \leq \Delta \frac{n}{\varphi(n)}.$$

Now any m, n with $m \geq 2$ we have

$$\begin{aligned} \frac{n}{\varphi(n)} \log m &\leq \frac{n}{\varphi(n)} \sum_{k=n}^{mn} \frac{1}{k} \\ &\leq \sum_{k=1}^{mn} \frac{1}{\varphi(k)} \\ &\leq \Delta \frac{mn}{\varphi(mn)}. \end{aligned}$$

Hence

$$\varphi(mn) \leq \frac{\Delta}{\log m} m\varphi(n).$$

For large m this shows that (4.1) holds.

Remark 5.8. The end of the proof of Example 5.7 actually establishes one direction of the following equivalence (the other direction is easier): $(\varphi(n))$ satisfies (URP) if and only if $(1/\varphi(n))$ is *regular*, i.e., if and only if there exists $C > 0$ such that

$$\frac{1}{\varphi(n)} \geq \frac{C}{n} \sum_{j=1}^{n} \frac{1}{\varphi(j)}.$$

Regular weight sequences arise also in the theory of Lorentz spaces.

Example 5.9. Now let us suppose, in addition, that $\varphi(n)/n^\delta$ is increasing for some choice of $\delta > 0$. We show how to make the preceding example reflexive.

Let $\psi(n) = \varphi(n)^{1+\delta} n^{-\delta}$. Then $\psi(n)/n$ is decreasing and $\psi(n)$ is increasing. Define X_ψ as in Example 5.7 for the function ψ. Let $\theta = (1+\delta)^{-1}$.

Let T denote Tsirelson space (cf. [2]). For our purposes it is only necessary to know that this space is reflexive,

$$\tfrac{1}{2}n \leq \|\sum_{j\in A} e_j\|_T \leq n \qquad \text{if } |A| = n$$

and

$$\|\sum_{j\in A} e_j^*\|_{T^*} \leq 2 \qquad \text{if } |A| = n \text{ and } n < A.$$

Now let $Y = [T, X_\psi]_\theta$ be the space obtained by complex interpolation. Since T is reflexive it follows from a result of Calderón [1] that Y is reflexive. Note that $Y^* = [T^*, X_\psi^*]_\theta$.

Now suppose $A \subset \mathbb{N}$ and $|A| = n$. Then

$$\|\sum_{j\in A} e_j\|_Y \leq n^{1-\theta}\|\sum_{j\in A} e_j\|_{X_\psi}^\theta \leq \varphi(n).$$

On the other hand if $n < A$ we have

$$\|\sum_{j\in A} e_j^*\|_{Y^*} \leq \|\sum_{j\in A} e_j^*\|_{T^*}^{1-\theta}\|\sum_{j\in A} e_j^*\|_{X_{\psi^*}}^\theta \leq 2(\frac{n}{\psi(n)})^\theta = 2\frac{n}{\varphi(n)}.$$

Hence for any A with $|A| = 2n$ we have

$$\|\sum_{j\in A} e_j\|_Y \geq \frac{\varphi(n)}{2}.$$

Thus (e_j) is democratic with fundamental function equivalent to φ. Now suppose (e_j^*) is democratic with constant Δ. Then

$$\|\sum_{j=1}^n e_j^*\|_Y \leq 2\Delta\frac{n}{\varphi(n)}$$

Now $Y^* = (T^*)^{1-\theta}(X_\psi^*)^\theta \subset Z := (\ell_\infty)^{1-\theta}(X_\psi^*)^\theta$ and so we have

$$\|\sum_{j=1}^n e_j^*\|_{Y^*} \geq \|\sum_{j=1}^n e_j^*\|_Z = \|\sum_{j=1}^n e_j^*\|_\infty^{1-\theta}\|\sum_{j=1}^n e_j^*\|_{X_\psi^*}^\theta = \|\sum_{j=1}^n e_j^*\|_{X_\psi^*}^\theta.$$

We deduce that

$$\|\sum_{j=1}^n e_j^*\|_{X_\psi^*} \leq (2\Delta\frac{n}{\varphi(n)})^{1/\theta} = (2\Delta)^{1/\theta}\frac{n}{\psi(n)}.$$

Hence by the argument presented in Example 5.7, we have that

$$\psi(m) \leq C_1\left(\frac{m}{n}\right)^\beta \psi(n) \qquad m > n$$

for some $\beta < 1$ and C_1. Now

$$\varphi(m) \leq C_1^{\frac{1}{1+\delta}} \left(\frac{m}{n}\right)^{\frac{\beta+\delta}{1+\delta}} \varphi(n) \qquad m > n.$$

This implies φ satisfies (4.1).

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Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

E-mail address: dilworth@math.sc.edu

Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA

E-mail address: nigel@math.missouri.edu

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Current address: Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

E-mail address: denka@banmatpc.math.bas.bg

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: temlyak@math.sc.edu