IMI
Preprint Series

Industrial Mathematics Institute

2001:17

Almost isometries of balls

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ALMOST ISOMETRIES OF BALLS

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Abstract. Let f be a bi-Lipschitz mapping of the Euclidean ball $B_{\mathbb{R}^n}$ into ℓ_2 with both Lipschitz constants close to one. We investigate the shape of $f(B_{\mathbb{R}^n})$. We give examples of such a mapping f, which has the Lipschitz constants arbitrarily close to one and at the same time has in the supremum norm the distance at least one from every isometry of $\mathbb{R}^n$.

1. Introduction

By the classical theorem of Mazur and Ulam, every surjective isometry f of two Banach spaces X and Y is affine. There are various possibilities how to slightly relax the isometry condition on f and still ask if f can be well approximated by an affine mapping (see [BL] for an exposition and literature on this subject). Here we will consider the case when both X and Y are Euclidean spaces and $f : B_X \to Y$ is a bi-Lipschitz mapping with both Lipschitz constants $1 + \varepsilon$ for some $0 < \varepsilon < 1$ (for exact definitions of an *ε-rigid mapping*, or of an *ε-quasi-isometry* see Section 2). If $\dim X = \dim Y = n$, then by a result of F. John [J], there is an isometry $T : X \to Y$ so that $\|f(x)-T(x)\| \le cn^{\frac{3}{2}}\varepsilon$ for $x \in B_X$, where c is an absolute constant. The estimation of the approximation error $\alpha(n, \varepsilon)$ was improved by Vestfrid [Ve] to $\alpha(n, \varepsilon) \le cn^{\frac{1}{2}}\varepsilon$. He proved also that in the general case when $n = \dim X \le \dim Y$ the approximation error is at most $cn^{\frac{1}{2}}\sqrt{\varepsilon}$. (If $\dim X < \dim Y$, the order of magnitude of the error has to be at least $\sqrt{\varepsilon}$. To see this, it is enough to take the mapping $f : [-1, 1] \to \mathbb{R}^2$ defined by $f(t) = (t, 0)$ if $t \in [-1, 0]$ and $f(t) = (t, t\sqrt{\varepsilon})$ if $t \in [0, 1]$. This mapping is ε-rigid and its distance from any affine mapping $T : \mathbb{R} \to \mathbb{R}^2$ is at least $\sqrt{\varepsilon}/8$.)

In Section 4 we give examples which show that the approximation error really does depend on the dimension of X, answering thus a

1991 *Mathematics Subject Classification.* 46C05, 30C65.

Key words and phrases. isometry, quasi-isometry, rigid mapping, approximate.

The author was partially supported by the grants GAAV-A1019103, and DEPSCOR-ONR N00014-99-1-0547.

question in [BL]. For example, for any $\varepsilon > 0$ we construct an ε-quasi-isometry $f : B_{\mathbb{R}^n} \to \mathbb{R}^n$ (n is about $\exp \frac{1}{\varepsilon}$) such that $f(B_{\mathbb{R}^{n/2}})$ contains an orthonormal basis of $\mathbb{R}^n$. This f has the distance at least $1/\sqrt{2}$ from every affine mapping of $\mathbb{R}^n$. Consequently, if we wish to write the approximation error in the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$, then $\varphi(n) \geq c \log n$ for some constant $c > 0$.

This is very much unlike the situation when both X and Y are Banach spaces of continuous functions on some compact metric spaces. Here, by a result of Lövblom [Lo], an ε-rigid mapping of B_X into Y can be approximated on $(1 - 8\varepsilon)B_X$ by an isometry within an error of 8ε.

We also investigate the shape of $f(B_{\mathbb{R}^n})$, if f is an ε-rigid mapping. In Proposition 3.1 and Proposition 3.2 an easy application of the theorem of Borsuk and Ulam shows that f can not "squeeze" $B_{\mathbb{R}^n}$ close to a space of dimension less than n: if Y is an affine space with $\dim Y < n$ then $f(B_{\mathbb{R}^n})$ is *not* contained in $Y + B(0, 1 - 4\sqrt{\varepsilon})$. In Proposition 3.4 we show a counterpart to Proposition 3.1: the convex hull K of an ε-rigid image of $B_{\mathbb{R}^n}$ can not fill up too much of $B_{\mathbb{R}^m}$ if $n < m$.

If Z is a closed linear subspace of a Hilbert space H, we denote by P_Z the orthogonal projection on Z. By $B_X(x, r)$ we denote the closed ball with the center at x and radius r in the Banach space X; $B_X^o(x, r)$ is the open ball. By $S_X(x, r)$ we denote the corresponding sphere. The unit ball with the center at zero is denoted by B_X. We reserve the notation $B_{\mathbb{R}^n}$ and $S_{\mathbb{R}^n}$ for the Euclidean ball and sphere. By $e_1, \dots, e_n$ we denote the standard orthonormal basis of $\mathbb{R}^n$. By $c, c_1, c_2, \dots$ we denote absolute constants, which may have different values even in the same formula.

2. Preliminaries

Let f be a mapping from an open subset U of a Banach space X into a Banach space Y. The local distortion of distances by f can be measured by the functions

$$D^+ f(x) = \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|},$$
$$D^- f(x) = \liminf_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

The following class of almost isometric mappings was introduced by F. John [J] (see [BL] for many of their properties).

Definition 2.1. Let $\varepsilon > 0$. A mapping f from an open subset U of a Banach space X into a Banach space Y is called an *ε-quasi-isometry* if it satisfies the following two conditions

(i) f is a local homeomorphism; *i.e.* every point $x \in U$ has an open neighborhood V such that f is a homeomorphism of V onto an open subset of Y.
(ii) f satisfies $(1+\varepsilon)^{-1} \le D^- f(x) \le D^+ f(x) \le 1+\varepsilon$ for every $x \in U$.

We will mostly work simply with bi-Lipschitz mappings which have the Lipschitz constants close to one:

Definition 2.2. Let $\varepsilon > 0$. A mapping f from a subset A of a Banach space X into a Banach space Y is called *ε-rigid* if $(1+\varepsilon)^{-1}\|x-y\| \le \|f(x)-f(y)\| \le (1+\varepsilon)\|x-y\|$ for all $x, y \in A$.

We will usually assume that $0 \in A$ and $f(0) = 0$. Also, we will often use the trivial observation that $1-\varepsilon \le (1+\varepsilon)^{-1} \le 1-\varepsilon/2$ for $0 < \varepsilon < 1$.

If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is ε-rigid then by the invariance of domains f is an ε-quasi-isometry (the invariance of domains says that if $V \subset \mathbb{R}^n$ is homeomorphic to an open set $U \subset \mathbb{R}^n$, then V itself is open in $\mathbb{R}^n$). The other way round, if X, Y are Banach spaces and $f : B^o_X(x,r) \to Y$ is an ε-quasi-isometry then f is ε-rigid on $B_X(x, r/(1+\varepsilon)^2)$ and $f(B_X(x,r)) \supset B_Y(f(x), r/(1+\varepsilon))$ (see *e.g.* [BL], p. 345).

It is an elementary, but useful fact that ε-rigid mappings almost preserve angles (see *e.g.* [BL], p. 349).

Lemma 2.3. *Let X be a Hilbert space, $0 < \varepsilon < 1$, $0 \in A \subset X$, and let $f : A \to X$ be ε-rigid and such that $f(0) = 0$. Then*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \tfrac{3}{2}\varepsilon(\|x-y\|^2 + \|x\|^2 + \|y\|^2)$$

for all $x, y \in A$.

Proof. Since f is ε-rigid, $|\|f(x)-f(y)\|^2 - \|x-y\|^2| \le 3\varepsilon\|x-y\|^2$ for $x, y \in A$. Hence

$$\begin{aligned} &2|\langle f(x), f(y)\rangle - \langle x, y\rangle| \\ &\quad \le |\|f(x)-f(y)\|^2 - \|x-y\|^2| + |\|f(x)\|^2 - \|x\|^2| + |\|f(y)\|^2 - \|y\|^2| \\ &\quad \le 3\varepsilon(\|x-y\|^2 + \|x\|^2 + \|y\|^2). \end{aligned}$$

□

The following lemma states that ε-rigid mappings almost preserve linearity for convex combinations. It is derived in [Ve] from a result of [Za].

Lemma 2.4. *Let X be a Hilbert space, $A \subset X$ be convex, $f : A \to X$ ε-rigid. Then for any $x_1, \ldots, x_n \in A$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ it holds*

$$\|f(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \leq \sqrt{2} \cdot \sqrt{\varepsilon} \max \|x_i - x_j\|.$$

This means in particular, that ε-rigid mappings of convex sets almost preserve the mid-points of line segments: $\|f(\frac{1}{2}(x+y)) - \frac{1}{2}(f(x) + f(y))\| \leq \sqrt{2}\sqrt{\varepsilon}\|x-y\|$ for $x, y \in A$.

Assume now that f is an ε-rigid mapping of a convex symmetric set A and $f(0) = 0$. Then f is almost antipodal; that is, $\|f(x) + f(-x)\| \leq 4\sqrt{2}\sqrt{\varepsilon}\|x\|$ for $x \in A$. Consequently, if $\lambda_i \in \mathbb{R}$ are such that $\sum_{i=1}^n |\lambda_i| = 1$, then

$$\begin{aligned}
&\|f(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \\
&\quad \leq \|f(\sum_{i=1}^n |\lambda_i|(x_i \cdot \operatorname{sgn} \lambda_i)) - \sum_{i=1}^n |\lambda_i| f(x_i \cdot \operatorname{sgn} \lambda_i)\| \\
&\qquad + \|\sum_{i=1}^n |\lambda_i| f(x_i \cdot \operatorname{sgn} \lambda_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \qquad (1)\\
&\quad \leq \sqrt{2} \cdot \sqrt{\varepsilon} \operatorname{diam} A + \sum_{i=1}^n |\lambda_i| \|f(x_i \cdot \operatorname{sgn} \lambda_i) - f(x_i) \cdot \operatorname{sgn} \lambda_i\| \\
&\quad \leq \sqrt{2} \cdot \sqrt{\varepsilon} \operatorname{diam} A + 4\sqrt{2} \cdot \sqrt{\varepsilon} \max \|x_i\| \\
&\quad \leq 3\sqrt{2}\sqrt{\varepsilon} \operatorname{diam} A.
\end{aligned}$$

This means that the image of a convex symmetric set by an ε-rigid mapping is again almost convex and almost symmetric.

Quasi-isometries preserve the mid-points of line segments with a smaller error $c\varepsilon$, instead of $c\sqrt{\varepsilon}$ for the ε-rigid mappings. The following lemma appears in [Ve] in a more general setting (f is a quasi-isometry between two Banach spaces), and with a rather involved proof. As we will use it only for quasi-isometries of Hilbert spaces, we provide here an elementary proof of this case.

Lemma 2.5. *Let $0 < a < 1$. There exists $\varepsilon_a > 0$ wit the following property. Let X be a Hilbert space, and let $f : B_X^o(0, 1+a) \to X$ be an*

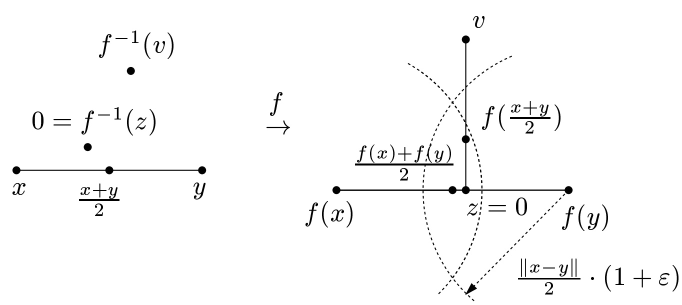

FIGURE 1. Illustration to the proof of Lemma 2.5.

ε-quasi-isometry for some $0 < \varepsilon \le \varepsilon_a$. Then

$$\|f(\tfrac{x+y}{2}) - \tfrac{f(x)+f(y)}{2}\| \le \tfrac{65}{a} \cdot \varepsilon \|x-y\|$$

for $x, y \in B_X$.

Proof. Let $0 < \varepsilon_a < \frac{1}{4}$ be such that

$$1 + \varepsilon + \tfrac{a}{2} \le (1+a)/(1+\varepsilon)^3 \tag{2}$$

for $0 < \varepsilon < \varepsilon_a$. By Theorem 14.7 of [BL],

(i) f is ε-rigid on $\frac{1+a}{(1+\varepsilon)^2} B_X \supset B_X$, and
(ii) $f(\frac{1+a}{(1+\varepsilon)^2} B_X) \supset B(f(0), \frac{1+a}{(1+\varepsilon)^3})$.

Let $x, y \in B_X$ be given. Let z be the orthogonal projection of $f(\frac{x+y}{2})$ on the line defined by $f(x)$ and $f(y)$. Since $f(\frac{x+y}{2}) \in B(f(x), \frac{\|x-y\|}{2}(1+\varepsilon)) \cap B(f(y), \frac{\|x-y\|}{2}(1+\varepsilon))$,

$$\begin{aligned} \|z - \tfrac{f(x)+f(y)}{2}\| &\le \tfrac{\|x-y\|}{2}(1+\varepsilon) - \tfrac{\|f(x)-f(y)\|}{2} \\ &\le \tfrac{\|x-y\|}{2}(1+\varepsilon - \tfrac{1}{1+\varepsilon}) \le \varepsilon\|x-y\|. \end{aligned} \tag{3}$$

Assume $z \ne f(\frac{x+y}{2})$; we will estimate $\|z - f(\frac{x+y}{2})\|$. To this end define

$$v = z + \frac{a}{4} \cdot (f(\tfrac{x+y}{2}) - z) \cdot \frac{\|x-y\|}{\|f(\frac{x+y}{2}) - z\|}.$$

From (3) it follows that $z \in \operatorname{conv}\{f(x), f(y)\}$. Since $f(x), f(y) \in B(f(0), 1+\varepsilon)$ we get by (2) and (ii) that $v \in f(\frac{1+a}{(1+\varepsilon)^2} B_X)$. Consequently, f is an ε-rigid mapping of the set $A = \{x, y, \frac{x+y}{2}, f^{-1}(z), f^{-1}(v)\}$. As we are interested only in estimating of distances of points in the set $f(A)$, we can by translation of A and of $f(A)$ assume that $0 = z = f^{-1}(z)$. Then, clearly, $\|f(x)\| \le \|f(x) - f(y)\| \le (1+\varepsilon)\|x-y\|$ and

$\|v - f(x)\| \le \|x-y\|(\frac{a}{4} + (1+\varepsilon))$. Since $\langle v, f(x)\rangle = \langle v, f(y)\rangle = 0$, by Lemma 2.3

$$|\langle f^{-1}(v), x\rangle| \le \tfrac{3}{2}\varepsilon(\|f(x)\|^2 + \|v\|^2 + \|f(x) - v\|^2) \le 6\varepsilon\|x-y\|^2,$$

and, similarly, $|\langle f^{-1}(v), y\rangle| \le 6\varepsilon\|x-y\|^2$. Hence $|\langle f^{-1}(v), \frac{x+y}{2}\rangle| \le 6\varepsilon\|x-y\|^2$, and again by Lemma 2.3

$$\begin{aligned}
|\langle v, f(\tfrac{x+y}{2})\rangle| &\le |\langle f^{-1}(v), \tfrac{x+y}{2}\rangle| \\
&\quad + \tfrac{3}{2}\varepsilon(\|f^{-1}(v)\|^2 + \|\tfrac{x+y}{2}\|^2 + \|f^{-1}(v) - \tfrac{x+y}{2}\|^2) \\
&\le 6\varepsilon\|x-y\|^2 + \tfrac{3}{2}\varepsilon\cdot 2(\|f^{-1}(v)\|^2 + \|\tfrac{x+y}{2}\|^2 + \|f^{-1}(v)\|\cdot\|\tfrac{x+y}{2}\|) \\
&\le 6\varepsilon\|x-y\|^2 + 3\varepsilon\|x-y\|^2((1+\varepsilon)^2a^2/16 + (1+\varepsilon)^4 + (1+\varepsilon)^3a/4) \\
&\le 16\varepsilon\|x-y\|^2.
\end{aligned}$$

By the definition of v

$$\|f(\tfrac{x+y}{2})\| = \langle v, f(\tfrac{x+y}{2})\rangle\cdot\tfrac{4}{a}\cdot\tfrac{1}{\|x-y\|} \le 16\varepsilon\|x-y\|^2\cdot\tfrac{4}{a}\cdot\tfrac{1}{\|x-y\|} \le \tfrac{64}{a}\cdot\varepsilon\cdot\|x-y\|,$$

and by (3)

$$\|f(\tfrac{x+y}{2}) - \tfrac{f(x)+f(y)}{2}\| \le \|z - \tfrac{f(x)+f(y)}{2}\| + \|f(\tfrac{x+y}{2}) - z\| \le \tfrac{65}{a}\varepsilon\|x-y\|.$$

□

Suppose that an ε-rigid mapping $f : B_{\mathbb{R}^n} \to \ell_2$ is well approximated by an affine mapping. Then f is well approximated by an isometry. This statement is used several times in [Ve]; for an easy reference we state it as a lemma.

Lemma 2.6. *Let $\varepsilon > 0$, $a > 0$ be such that $a+\varepsilon < 1$. Let $f : B_{\mathbb{R}^n} \to \ell_2$ be ε-rigid and $T : \mathbb{R}^n \to \ell_2$ linear such that $\|f(x) - T(x)\| \le a$ for all $x \in B_{\mathbb{R}^n}$. Then there exists an isometry $\mathbb{R}^n \to \ell_2$ so that $\|f(x) - S(x)\| \le \varepsilon + 2a$ for all $x \in B_{\mathbb{R}^n}$.*

Proof. Let $u_1, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$, $v_1, \dots, v_m$ an orthonormal basis of $T(\mathbb{R}^n)$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m > 0$ so that $T(u_i) = \lambda_i v_i$ for $i = 1, \dots, m$ and $T(u_i) = 0$ and $\lambda_i = 0$ for $i > m$. Then

$$\begin{aligned}
\lambda_i &= \|T(u_i)\| \le \|f(u_i)\| + a \le 1 + \varepsilon + a \quad \text{and} \\
\lambda_i &= \|T(u_i)\| \ge \|f(u_i)\| - a \ge 1 - \varepsilon - a,
\end{aligned}$$

hence $1 + \varepsilon + a \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 1 - \varepsilon - a > 0$; in particular, $m = n$. Define S by $S(u_n) = v_n$. Then $\|S - T\| = \max_{i\in\{1,\dots,n\}} |1 - \lambda_i| \le a + \varepsilon$, and the lemma follows from the triangle inequality. □

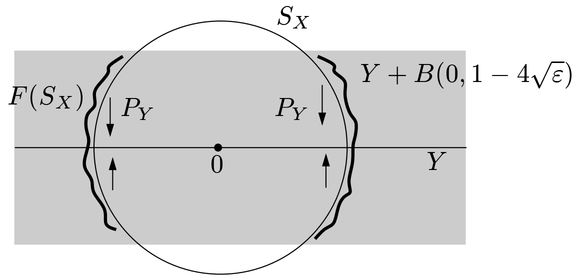

FIGURE 2. Illustration to the proof of Proposition 3.1.

3. ε-RIGID MAPPINGS AND LINEAR SUBSPACES

Let X be a Banach space and $A \subset X$; let $k \in \mathbb{N}$. Recall, that the Kolmogorov k-diameter $d_k(A, X)$ of A expresses how well can A be approximated by k-dimensional subspaces of X:

$$d_k(A, X) = \inf_{X_k} \sup_{x \in A} \inf_{y \in X_k} \|x - y\|,$$

the left-most infimum being taken over all k-dimensional subspaces X_k of X. The sum of a linear subspace and of a ball is a convex set, hence $d_k(A, X) = d_k(\operatorname{conv} A, X)$ (for other properties of the Kolmogorov diameter see *e.g.* [Pi]).

First we observe that an ε-rigid mapping f can not squeeze the unit ball of a k-dimensional Hilbert space inside of a small neighborhood of a space with dimension $l < k$. We will actually show that the Kolmogorov l-diameter of $f(B_{\mathbb{R}^k})$ is almost one.

Proposition 3.1. *Let* $0 < \varepsilon < 1$ *and* $f : B_{\mathbb{R}^n} \to \ell_2$ *be* ε*-rigid,* $f(0) = 0$*. Let* $X \subset \mathbb{R}^n$*,* $Y \subset \ell_2$ *with* $\dim Y < \dim X$*. Then* $f(B_X)$ *is* not *contained in* $Y + B_{\ell_2}(0, 1 - 4\sqrt{\varepsilon})$*.*

Proof. Suppose that

$$f(B_X) \subset U := Y + B_{\ell_2}(0, 1 - 4\sqrt{\varepsilon}).$$

Assuming this, we will construct a continuous antipodal mapping $\Phi : S_X \to Y$ such that $\Phi(x) \neq 0$ for all $x \in S_X$, which will contradict the Borsuk-Ulam theorem. For $x \in S_X$ put $F(x) = \frac{1}{2}(f(x) - f(-x))$ and define $\Phi = P_Y \circ F$. The mapping F is antipodal, as $F(-x) = \frac{1}{2}(f(-x) - f(x)) = -F(x)$, hence Φ is antipodal as well. By the remark after Lemma 2.4

$$\begin{aligned} \|F(x)\| &= \|f(x) - \tfrac{1}{2}(f(-x) + f(x))\| \geq \|f(x)\| - 2\sqrt{2}\sqrt{\varepsilon} \\ &\geq 1 - \varepsilon - 2\sqrt{2}\sqrt{\varepsilon} > 1 - 4\sqrt{\varepsilon}. \end{aligned} \tag{4}$$

Since U is convex and symmetric, $F(S_X) \subset U$, and

$$\|P_Y(F(x))\| \geq \|F(x)\| - (1 - 4\sqrt{\varepsilon}). \tag{5}$$

By (4) and (5)

$$\|\Phi(x)\| = \|P_Y(F(x))\| > 1 - 4\sqrt{\varepsilon} - (1 - 4\sqrt{\varepsilon}) = 0.$$

□

The midpoints of line segments are for ε-quasi-isometries by Lemma 2.5 preserved with the error $c\varepsilon$ instead just $c\sqrt{\varepsilon}$ as it was for ε-rigid mappings. This enables slightly improve Proposition 3.1; the proof is the same.

Proposition 3.2. *Let $0 < a < 1$. There exists $\varepsilon_a > 0$ with the following property. Let $f : B^o_{\mathbb{R}^n}(0, 1 + a) \to \mathbb{R}^n$, $f(0) = 0$ be an ε-quasi isometry for some $0 < \varepsilon \leq \varepsilon_a$. Suppose $X, Y \subset \mathbb{R}^n$ with $\dim Y < \dim X$. Then $f(B_X)$ is* not *contained in $Y + B_{\mathbb{R}^n}(0, 1 - \frac{140}{a} \cdot \varepsilon)$.*

To prove a counterpart to Proposition 3.1, we will need the following version of the theorem of Bartle and Graves.

Theorem 3.3. *Let X, Y be Banach spaces, $T : X \to Y$ continuous, linear and surjective and $K \subset X$ closed and convex. Then there exists a continuous mapping $f : T(K) \to K$ so that $T(f(y)) = y$ for all $y \in T(K)$. Moreover, if K is symmetric, f can be chosen so that $f(y) = -f(-y)$ for all $y \in T(K)$.*

Proof. (Sketch) We can simply follow the proof in ([BP], p. 86). Let Φ be the inverse of T restricted to K. Then $\Phi : T(K) \to 2^X$ is a complete convex lower semi-continuous mapping. By Michael's theorem Φ admits a continuous selection f. If K is moreover symmetric, we replace $f(y)$ by $\frac{1}{2}(f(y) - f(-y))$. □

Next we prove that the convex hull K of an ε-rigid image of a k-dimensional unit ball can not fill up too much of an l-dimensional unit ball if $l > k$. Namely, the maximal inscribed ball of the projection $P_Y(K)$ onto any Y with $\dim Y = l > k$ has radius only $c\sqrt{\varepsilon}$. Notice however, that this does not mean that K is contained in a small neighborhood of a k-dimensional space. This follows from Example 4.1.

Proposition 3.4. *Let $0 < \varepsilon < \frac{1}{2}$ and $f : B_{\mathbb{R}^n} \to \ell_2$ be ε-rigid, $f(0) = 0$. Let $X \subset \mathbb{R}^n$, $Y \subset \ell_2$ with $\dim Y = \dim X + 1$, and $K = \operatorname{sym\,conv} f(B_X)$. Then $\max\{r : B_Y(0, r) \subset P_Y(K)\} < 30\sqrt{\varepsilon}$.*

Proof. Assume that $B_Y(0, 30\sqrt{\varepsilon}) \subset P_Y(K)$. As in the proof of Proposition 3.1, we will construct under this assumption a continuous antipodal mapping $\Phi : S_Y(0, 30\sqrt{\varepsilon}) \to X$ such that $\Phi(y) \neq 0$ for all

$y \in S_Y(0, 30\sqrt{\varepsilon})$. This will contradict the Borsuk-Ulam theorem. The mapping $f^{-1} : f(B_X) \to X$ is $(1+\varepsilon)$-Lipschitz; by the theorem of Kirszbraun (see *e.g.* [BL], p. 19) it can be extended to a $(1+\varepsilon)$-Lipschitz mapping $\varphi : \ell_2 \to X$. For $v \in \ell_2$ put $F(v) = \frac{1}{2}(\varphi(v) - \varphi(-v))$; clearly, F is antipodal. Let $v \in K$. By (1), there exists $x \in B_X$ so that $\|f(x) - v\| \le 6\sqrt{2}\sqrt{\varepsilon}$. Since $\|f(x)\| \le (1+\varepsilon)\|x\|$,

$$\|x\| \ge (\|v\| - 6\sqrt{2}\sqrt{\varepsilon})(1+\varepsilon)^{-1} \ge \tfrac{2}{3}\|v\| - 4\sqrt{2}\sqrt{\varepsilon}.$$

By the definition of φ we have $\|x - \varphi(v)\| \le (1+\varepsilon)\|f(x) - v\|$, hence

$$\begin{aligned} \|\varphi(v)\| &\ge \|x\| - (1+\varepsilon)\|v - f(x)\| \\ &\ge \tfrac{2}{3}\|v\| - 13\sqrt{2}\sqrt{\varepsilon}. \end{aligned} \tag{6}$$

By Theorem 3.3, there exists a continuous mapping $\psi : P_Y(K) \to K$ such that $P_Y(\psi(y)) = y$ and $\psi(y) = -\psi(-y)$ for all $y \in P_Y(K)$. As ψ is a selection from the inverse of an orthogonal projection, it is also $\|\psi(y)\| \ge \|y\|$. Define $\Phi : P_Y(K) \to X$ by $\Phi = \varphi \circ \psi$. Let $y \in S_Y(0, 30\sqrt{\varepsilon}) \subset P_Y(K)$. Then by (6)

$$\begin{aligned} \|\Phi(y)\| = \|\varphi(\psi(y))\| &\ge \tfrac{2}{3}\|\psi(y)\| - 13\sqrt{2}\sqrt{\varepsilon} \\ &\ge \tfrac{2}{3}\|y\| - 13\sqrt{2}\sqrt{\varepsilon} > 0, \end{aligned} \tag{7}$$

and this contradicts the Borsuk-Ulam theorem. □

If $T : \mathbb{R}^n \to \ell_2$ is affine, then, clearly, the graph of T is contained in an n-dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$. If some $f : B_{\mathbb{R}^n} \to \ell_2$ is well approximated by an affine mapping, then the graph of f is contained in a small neighborhood of an n-dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$. In Lemma 3.5 we observe that the converse also holds. If the graph of a mapping $f : B_{\mathbb{R}^n} \to 2B_{\ell_2}$ is contained in a small neighborhood of an n-dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$, then f is well approximated by an affine mapping.

Lemma 3.5. *Let $f : B_{\mathbb{R}^n} \to \ell_2$ be a mapping with $\|f(x)\| \le 2$ for $x \in B_{\mathbb{R}^n}$. Suppose there is an n-dimensional subspace $Z \subset \mathbb{R}^n \oplus \ell_2$ and $0 < \delta < \frac{1}{2}$ such that the graph of f is contained in $Z + B_{\mathbb{R}^n \oplus \ell_2}(0, \delta)$. Then there is a linear mapping $T : \mathbb{R}^n \to \ell_2$ so that $\|T(x) - f(x)\| \le 7\delta$ for all $x \in B_{\mathbb{R}^n}$.*

Proof. Let $P = P_{\mathbb{R}^n}$ be the orthogonal projection on $\mathbb{R}^n$. We can assume that $P : Z \to \mathbb{R}^n$ is a bijection; this can be achieved by an arbitrarily small perturbation of Z. Put $S = P^{-1}$; S has necessarily the form $S(x) = (x, T(x))$ with T linear. Choose orthonormal bases

$\{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ and $\{v_1, \dots, v_n\}$ of Z, so that $T(u_i) = \lambda_i v_i$ for some $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Choose $y \in \mathbb{R}^n$ so that

$$\tfrac{1}{2} > \delta \geq \operatorname{dist}\left((u_1, f(u_1)), Z\right) = (\|u_1 - y\|^2 + \|T(y) - f(u_1)\|^2)^{\frac{1}{2}}.$$

If $\langle z, u_1 \rangle < \frac{1}{2}$ for some $z \in \mathbb{R}^n$, then $\|z - u_1\| \geq \frac{1}{2}$. Hence $\langle y, u_1 \rangle \geq \frac{1}{2}$, and

$$\tfrac{1}{2} \geq \|T(y) - f(u_1)\| \geq \|T(y)\| - \|f(u_1)\| \geq \lambda_1 \langle y, u_1 \rangle - 2.$$

This implies that $\|T\| = \lambda_1 \leq 5$, and $\|S\| \leq 6$.

Let $x \in B_{\mathbb{R}^n}$; denote $F(x) = (x, f(x))$. For $y = P(P_Z(F(x)))$ it holds

$$\begin{aligned} \|x - y\| &= \|P(F(x)) - P(S(y))\| \leq \|P\| \cdot \|F(x) - S(y)\| \\ &= \|F(x) - P_Z(F(x))\| \leq \delta. \end{aligned}$$

Hence

$$\begin{aligned} \|T(x) - f(x)\| &= \|S(x) - F(x)\| \leq \|S(x) - S(y)\| + \|S(y) - F(x)\| \\ &\leq \|S\|\delta + \delta \leq 7\delta. \end{aligned}$$

□

If f is an ε rigid mapping, then by an elementary computation (which we perform below) the mapping $F(x) = \frac{1}{\sqrt{2}}(x, f(x))$ is 2ε-rigid. Suppose f is not well approximated by affine mappings, for example, $f(0) = 0$ and $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq \delta > 0$ for all linear mappings T. Then by Lemma 3.5, the Kolmogorov n-diameter of $F(B_{\mathbb{R}^n})$ is large, namely $d_n(F(B_{\mathbb{R}^n}), \ell_2) \geq \delta/7$.

Lemma 3.6. *Let $A \subset \ell_2$ and $f : A \to \ell_2$ ε-rigid for some $\varepsilon > 0$. Let $K > 0$ and $F : A \to \ell_2$ be the mapping which gives each $x \in A$ its image in the graph of $K \cdot f$; that is, $F(x) = (x, Kf(x))$ (here we write $\ell_2 = \ell_2 \oplus \ell_2$). Then for all $x, y \in A$*

$$(\sqrt{1 + K^2} - \varepsilon K)\|x - y\| \leq \|F(x) - F(y)\| \leq (\sqrt{1 + K^2} + \varepsilon K)\|x - y\|.$$

Proof. If $x \neq y$, then

$$\frac{\|F(x) - F(y)\|^2}{\|x - y\|^2} = 1 + K^2 \frac{\|f(x) - f(y)\|^2}{\|x - y\|^2},$$

and

$$(1 - \varepsilon)^2 \leq \frac{\|f(x) - f(y)\|^2}{\|x - y\|^2} \leq (1 + \varepsilon)^2.$$

Moreover

$$\sqrt{1 + K^2} - \varepsilon K \leq \sqrt{1 + K^2(1 - \varepsilon)^2} \text{ and } \sqrt{1 + K^2(1 + \varepsilon)^2} \leq \sqrt{1 + K^2} + \varepsilon K.$$

□

4. A quasi-isometry can be far from all isometries

Consider the following example by F. John [J] (see also [BL], p. 352). Let $0 < \varepsilon < 1$. The mapping h of the unit disc $B_{\mathbb{R}^2}$ onto itself defined in the polar coordinates by $h(r,\varphi) = (r, \varphi + \varepsilon \log r)$ for $r > 0$ and by $h(0) = 0$ is an ε-quasi-isometry; it actually satisfies $(1+\varepsilon)^{-1}\|x-y\| \le \|h(x)-h(y)\| \le (1+\varepsilon)\|x-y\|$ for all $x, y \in B_{\mathbb{R}^2}$. If we define h outside of the unit disc by $h(x) = x$, the above inequality holds for all $x, y \in \mathbb{R}^2$. This can be seen by direct checking; also, it follows immediately from Lemma 2 of [IP] applied to both h and the inverse of h. In the supremum norm, h can be well approximated by the identity. It rotates each $x \in B_{\mathbb{R}^2}$ around the origin by an angle $\varepsilon \log(\|x\|)$; close to the origin this changes a lot.

We will use h to construct an ε-quasi-isometry f of $B_{\mathbb{R}^{2n}}$ onto itself (n is about $\exp \frac{1}{\varepsilon}$) so that the image of $B_{\mathbb{R}^n}$ nearly contains the unit ball $B_{\ell_1^{2n}}$. As any affine mapping carries $\mathbb{R}^n$ to an affine subspace of dimension at most n, the mapping f can not be well approximated by an isometry.

Theorem 4.1. *Let $0 < \varepsilon < 1$ be given. There exists $n \in \mathbb{N}$ and a norm preserving ε-quasi-isometry f of $\mathbb{R}^{2n}$ onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^{2n}$, and $f(B_{\mathbb{R}^n})$ contains an orthonormal basis of $\mathbb{R}^{2n}$. Consequently,*

(i) $d_k(f(B_{\mathbb{R}^n}), \ell_2^{2n}) \ge \sqrt{1 - \frac{k}{2n}}$ *for* $1 \le k \le 2n$;
(ii) *if* $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *is affine, then* $\sup_{x \in B_{\mathbb{R}^{2n}}} \|T(x) - f(x)\| \ge \frac{1}{\sqrt{2}}$, *and*
(iii) $B_{\ell_1^{2n}} \subset f(B_{\mathbb{R}^n}) + B_{\mathbb{R}^{2n}}(0, 2\sqrt{\varepsilon})$.

Proof. We can assume that ε is of the form $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$, where $K \in \mathbb{N}$ is large enough, and put $n = 2^K$. We write $\mathbb{R}^{2n}$ as $\mathbb{R}^n \oplus \mathbb{R}^n$. Let $e_1, \dots, e_n$ be the standard orthonormal basis of the first copy of $\mathbb{R}^n$, and let $e_{n+1}, e_{n+2}, \dots, e_{2n}$ be the standard orthonormal basis of the second copy of $\mathbb{R}^n$. Let $u_1, \dots, u_n$ be the orthonormal basis of the first $\mathbb{R}^n$ which corresponds to the columns of the Hadamard matrix; that is, each u_j is of the form $u_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_i$, where $\varepsilon_{i,j} \in \{1, -1\}$ are suitably chosen. Similarly, let $v_1, \dots, v_n$ be an orthonormal basis of the second $\mathbb{R}^n$ for which $v_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_{n+i}$.

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined above; let $g = e^{\pi i/2} h$ be h composed with the rotation by $\pi/2$ around the origin. Then g rotates by $\pi/2$ all $z \in \mathbb{R}^2$ with $\|z\| \ge 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with

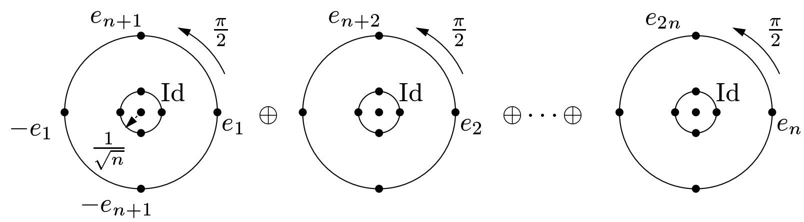

FIGURE 3. Illustration to the proof of Theorem 4.1.

$\|z\| = 1/\sqrt{n}$, as

$$\frac{\pi}{2} + \varepsilon \log \frac{1}{\sqrt{n}} = \frac{\pi}{2} + \frac{\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{1}{\sqrt{2^K}} = 0.$$

Below we will consider g written in the Cartesian coordinates. Now we will write $\mathbb{R}^{2n}$ as the ℓ_2-sum of n copies of $\mathbb{R}^2$:

$$\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 = \operatorname{span}\{e_1, e_{n+1}\} \oplus \operatorname{span}\{e_2, e_{n+2}\} \oplus \cdots \oplus \operatorname{span}\{e_n, e_{2n}\}.$$

We define f "coordinate-wise": if $x = \sum_{i=1}^n (x_i e_i + x_{n+i} e_{n+i})$ then

$$\begin{aligned} f(x) &= f((x_1, x_{n+1}), (x_2, x_{n+2}), \ldots, (x_n, x_{2n})) \\ &= (g(x_1, x_{n+1}), g(x_2, x_{n+2}), \ldots, g(x_n, x_{2n})). \end{aligned}$$

Since g preserves the norm and $g(z) = -g(-z)$ for $z \in \mathbb{R}^2$, it holds $\|f(x)\| = \|x\|$ and $f(x) = -f(-x)$ for $x \in \mathbb{R}^{2n}$. Since g is a bi-Lipschitz mapping of $\mathbb{R}^2$ onto itself, f is a bi-Lipschitz mapping of $\mathbb{R}^{2n}$ onto itself and the Lipschitz constants are the same; that is, $(1+\varepsilon)^{-1}\|x-y\| \le \|f(x) - f(y)\| \le (1+\varepsilon)\|x-y\|$ for all $x, y \in \mathbb{R}^{2n}$. The projection of e_j, $j \in \{1, \ldots n\}$ on each of the 2-dimensional blocks spanned by $\{e_k, e_{n+k}\}$ is either e_j itself (if $j = k$), or zero. As $g(0) = 0$ and g rotates by $\pi/2$ on the unit circle, we have $f(e_j) = e_{n+j}$ for $j = 1, \ldots, n$. The projection $p_k(u_j)$ of u_j on each of the 2-dimensional blocks spanned by $\{e_k, e_{n+k}\}$ is $p_k(u_j) = \frac{1}{\sqrt{n}} \varepsilon_{k,j} e_k$, hence $\|p_k(u_j)\| = \frac{1}{\sqrt{n}}$. Therefore $g(p_k(u_j)) = p_k(u_j)$ for $k = 1, \ldots, n$ and $f(u_j) = u_j$ for each $j = 1, \ldots, n$. Consequently, as $f(x) = -f(-x)$, the image of the first copy of $\mathbb{R}^n$ contains (both plus and minus) the orthonormal basis $Q = \{u_1, u_2, \ldots, u_n, e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n}$. For completeness, let us mention, that this way we also obtain that $f(e_{n+j}) = -e_j$ and $f(v_j) = v_j$ for $n = 1, \ldots, n$.

Since $\pm Q \subset f(B_{\mathbb{R}^n})$, and $B_{\ell_1^{2n}} = \operatorname{conv} \pm Q$, the statement (i) follows from the estimate $d_k(B_{\ell_1^{2n}}, \ell_2^{2n}) = \sqrt{1 - \frac{k}{2n}}$, $k \in \{1, \ldots, 2n\}$ for the Kolmogorov diameter of the ball of ℓ_1^{2n} (see e.g. [T], p. 237). In particular, since $\pm Q$ is symmetric, if Z is an n-dimensional affine subspace of

$\mathbb{R}^{2n}$, then there exists $q \in \pm Q$ so that $\operatorname{dist}(Z, q) \geq 1/\sqrt{2}$. This implies (ii), as $Z = T(\mathbb{R}^n)$ is an at most n-dimensional affine subspace of $\mathbb{R}^n$. The statement (iii) follows from Lemma 2.4, since $\pm Q \subset f(B_{\mathbb{R}^n})$. □

Let $f : B_{\mathbb{R}^n} \to \mathbb{R}^n$ be an ε-quasi-isometry for some $0 < \varepsilon < 1$. Denote by $\alpha(f) = \inf_T \sup_{x \in B_{\mathbb{R}^n}} \|T(x) - f(x)\|$, where the infimum is taken over all affine mappings $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\alpha(n, \varepsilon) = \sup_f \alpha(f)$, the supremum being taken over all f as above. By [J] and [Ve], $\alpha(n, \varepsilon) \leq c\sqrt{n}\varepsilon$. If we similarly define $\beta(n, \varepsilon)$ for ε-rigid mappings, then by [Ve], $\beta(n, \varepsilon) \leq c\sqrt{n}\sqrt{\varepsilon}$. Theorem 4.1 implies, that if we wish to write $\alpha(n, \varepsilon)$ in the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$, then it holds $\varphi(n) \geq c \log n$, where $c > 0$ is a suitable constant. Indeed, if $n \in \mathbb{N}$, choose $K \in \mathbb{N}$ so that $2^{K+1} \leq n < 2^{K+2}$, that is, $K = \lfloor \log n / \log 2 \rfloor - 1$. In the proof of Theorem 4.1 we constructed an ε-quasi-isometry $f : \mathbb{R}^{2^{K+1}} \to \mathbb{R}^{2^{K+1}}$, with $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$, so that $\alpha(f) = 1/\sqrt{2}$. If we write $\mathbb{R}^n = \mathbb{R}^{2^{K+1}} \oplus \mathbb{R}^{n-2^{K+1}}$ and define $F : \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, y) = (f(x), y)$, then F is also an ε-quasi-isometry with $\alpha(f) = 1/\sqrt{2}$. Hence

$$\frac{1}{\sqrt{2}} \leq \varphi(n)\varepsilon = \varphi(n) \cdot \frac{\pi}{\log 2} \cdot \frac{1}{\lfloor \log n / \log 2 \rfloor - 1},$$

and $\varphi(n) \geq c \log n$ for a suitable $c > 0$. Similarly, if we wish to write $\beta(n, \varepsilon)$ in the form $\beta(n, \varepsilon) = \psi(n)\sqrt{\varepsilon}$, then it holds $\psi(n) \geq c \log^{\frac{1}{2}} n$, where $c > 0$ is a suitable constant. This shows that the approximation error for near-isometries which was estimated in [ATV] also does depend on the dimension.

A natural approach how to try to approximate an ε-quasi-isometry f defined on $B_{\mathbb{R}^n}$ by a linear mapping T is to fix an orthonormal basis of $\mathbb{R}^n$ (for example $\{e_1, \ldots, e_n\}$), and put $T(e_i) = \frac{1}{2}(f(e_i) - f(-e_i))$ for $i = 1, \ldots, n$. This is basically used in both [J] and [Ve]. Again, if we wish the approximation error to be of the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$ with $\varphi(n)$ as small as possible, the best this approach can give to us is $\varphi(n) = c\sqrt{n}$, as was achieved in [Ve].

Lemma 4.2. *Let n be large enough. There exists an isometry S of $\mathbb{R}^n$ with $\|S - \mathrm{Id}\| = 2$ and $\frac{8}{\sqrt{n}}$-quasi-isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$, so that $\|S(x) - f(x)\| \leq \frac{2}{\sqrt{n}}$ for $x \in B_{\mathbb{R}^n}$ and at the same time $f(\pm e_i) = \pm e_i$ for $i = 1, \ldots, n$.*

Moreover, if $n = 2^k$ for some $k \in \mathbb{N}$, and $u_1, \ldots, u_n$ is the orthonormal basis of $\mathbb{R}^n$ which corresponds to the columns of the Hadamard matrix then $f(\pm u_1) = \mp u_1$, and $f(\pm u_i) = \pm u_i$ for $i = 2, \ldots, n$.

Proof. Let $v = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i$. Then $\|v\| = 1$ and v is "almost orthogonal" to all e_i's; that is, $\langle v, e_i \rangle = \frac{1}{\sqrt{n}}$ for all $i = 1, \dots, n$. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be the isometry which coincides with the identity on $\operatorname{Ker} v$ and $S(v) = -v$; that is, $S(x) = x - 2\langle x, v \rangle v$. Let φ be the function supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$, for which $\varphi(0) = \frac{2}{\sqrt{n}}$ and φ is linear on $[-\frac{1}{2}, 0]$ and on $[0, \frac{1}{2}]$. Define $\varphi_i^+ : \mathbb{R}^n \to \mathbb{R}$ by $\varphi_i^+(x) = \varphi(\|x - e_i\|)$, and, similarly, $\varphi_i^-(x) = -\varphi(\|x + e_i\|)$. As the distances of different $\pm e_i$'s are at least $\sqrt{2}$, the functions φ_i are disjointly supported. Consequently, as the function φ is $\frac{4}{\sqrt{n}}$-Lipschitz, the function $\Phi = \sum_{i=1}^{n} (\varphi_i^+ + \varphi_i^-)$ is $\frac{4}{\sqrt{n}}$-Lipschitz as well, with $|\Phi| \le \frac{2}{\sqrt{n}}$. For $x \in B_{\mathbb{R}^n}$ put $f(x) = S(x) + \Phi(x)v$. Then f is $\frac{8}{\sqrt{n}}$-rigid and $\|S(x) - f(x)\| \le |\Phi(x)| \le \frac{2}{\sqrt{n}}$ for $x \in \mathbb{R}^n$. Moreover, $f(0) = 0$ and $f(\pm e_i) = \pm e_i - 2\langle \pm e_i, v \rangle v + \varphi_i^{\pm}(e_i) v = \pm e_i$.

Suppose that $n = 2^k$. We can assume that $u_1 = v$. As $S = \mathrm{Id}$ on $\operatorname{Ker} v$ and $\|u_i \pm e_j\| \ge \sqrt{(n-1)/n} > 1/2$, $f(\pm u_i) = \pm u_i$ for $i = 2, \dots, n$. □

To present a modification of Theorem 4.1 we recall a few basic facts about permutations. A *permutation* p on a finite set Ω is a bijection of Ω onto itself. Each permutation can be decomposed uniquely, except for order, into disjoint cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 4 & 6 & 7 \end{pmatrix} = (1,2,3)(4,5)(6)(7) = (1,2,3)(4,5);$$

in the last expression the single point cycles (that is, the fixed points of the permutation) are omitted. A transposition is a permutation which consists of one cycle of length two; all the other cycles have length one. Every permutation can be written as a composition of (enough many) transpositions. Each permutation can be composed from four permutations each of which consist only of *disjoint* transpositions (and of single point cycles). For convenience we include a simple proof of this.

Lemma 4.3. *Let p be permutation on a finite set Ω. Then $p = p_4 \circ p_3 \circ p_2 \circ p_1$, where each of the permutations $p_1, \dots, p_4$ consists of* disjoint *cycles of length at most two.*

Proof. We can assume that p is a cycle. For cycles of length up to five it holds: $(1,2,3) = (2,3) \circ (1,2)$; $(1,2,3,4) = (2,4) \circ [(1,2)(3,4)]$; and $(1,2,3,4,5) = (4,5) \circ (2,4) \circ [(1,2)(3,4)]$.

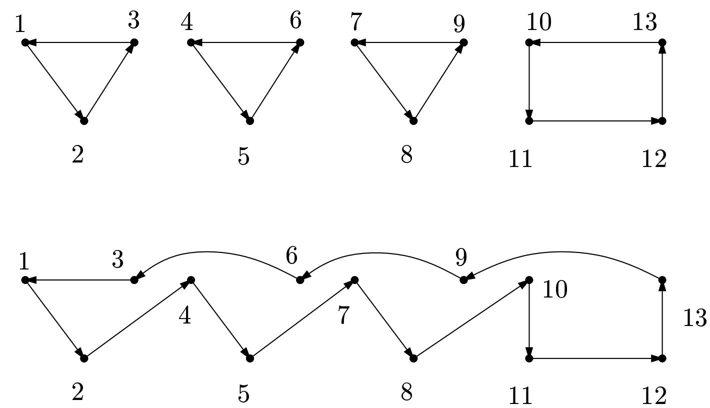

FIGURE 4. The permutations p' and p for $n = 13$.

If $|\Omega| = n > 5$, we write $n = 3k + l$, where $k \in \mathbb{N}$ and $l \in \{3, 4, 5\}$, and put

$$p' = (1, 2, 3)(4, 5, 6)(7, 8, 9) \dots (3k - 2, 3k - 1, 3k)(3k + 1, \dots, 3k + l),$$
$$p_4 = (3, 4)(6, 7) \dots (3k, 3k + 1).$$

By the special cases mentioned above, $p' = p_3 \circ p_2 \circ p_1$, where each of the permutations $p_1, \dots, p_3$ consists of disjoint cycles of length at most two. The permutation p_4 joins the k triangles and one l-gon of the permutation p' into a single cycle:

$$p_4 \circ p' = (1; 2, 4, 5, 7, \dots, 3k-1, 3k+1; 3k+2, \dots 3k+l; 3k, 3(k-1), \dots, 3).$$

By denoting the elements of Ω successively (according to the cycle p) by $1, 2, 4, 5, 7, \dots, 3$ we get that $p = p_4 \circ p_3 \circ p_2 \circ p_1$. $\square$

Theorem 4.4. *Let $0 < \varepsilon < 1$ be given. There exist $n \in \mathbb{N}$ and two orthonormal bases $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ with the following property. Let p be a permutation on $\{1, 2, \dots, n\}$, and let $\alpha_i \in \{-1, 1\}$. There exists an ε-quasi-isometry f of $\mathbb{R}^n$ onto itself so that $f(\pm e_i) = \pm\alpha_i e_{p(i)}$ and $f = \mathrm{Id}$ on $\{0, \pm u_1, \dots, \pm u_n\}$.*

Proof. Choose $K \in \mathbb{N}$ so that $(1 + \frac{2\pi}{\log 2} \cdot \frac{1}{K})^6 \le 1 + \varepsilon$ and put $n = 2^{K+2}$. Let $e_1, \dots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Let $u_1, \dots, u_n$ be the orthonormal basis of $\mathbb{R}^n$ which corresponds to the columns of the Hadamard matrix; that is, each u_j is of the form $u_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_i$, where $\varepsilon_{i,j} \in \{1, -1\}$ are suitably chosen. We will prove two special cases of the theorem:

(A) Suppose p consists of disjoint cycles of length at most two. Then there exists a norm preserving $(\frac{\pi}{\log 2} \cdot \frac{1}{K})$-quasi-isometry f of $\mathbb{R}^n$

onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, $f(e_i) \in \{\pm e_{p(i)}\}$ and $f = \mathrm{Id}$ on $\{u_1, \dots, u_n\}$.

(B) Suppose that $|\{i : \alpha_i = -1\}|$ is even. Then there exists a norm preserving $(\frac{2\pi}{\log 2} \cdot \frac{1}{K})$-quasi-isometry f of $\mathbb{R}^n$ onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, $f(e_i) = \alpha_i e_i$ and $f = \mathrm{Id}$ on $\{u_1, \dots, u_n\}$.

To get the general case, we write as in Lemma 4.3 $p = p_4 \circ p_3 \circ p_2 \circ p_1$, where each of the permutations $p_1, \dots, p_4$ consists of disjoint cycles of length at most two. For each $j \in \{1, \dots, 4\}$, let f_j be the quasi-isometry which exists by (A) for the permutation p_j. The quasi-isometry $\tilde{f} = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies $\tilde{f}(e_i) = \beta_i e_{p(i)}$ for some $\beta_i \in \{-1, 1\}$, and $\tilde{f} = \mathrm{Id}$ on $\{u_1, \dots, u_n\}$. If $|\{i : \alpha_i \neq \beta_i\}|$ is even, there exists by (B) a quasi-isometry f_5 so that $f = f_5 \circ \tilde{f}$ satisfies the conclusion of the theorem. Suppose $|\{i : \alpha_i \neq \beta_i\}|$ is odd; we can assume that $\alpha_1 \neq \beta_1$. By (B) there exists a quasi-isometry f_5 so that $f_5 \circ \tilde{f}$ satisfies the conclusion of the theorem but for $(f_5 \circ \tilde{f})(e_1) = -\alpha_1 e_1$. By Lemma 4.2 (with the bases $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ interchanged), there exists a quasi-isometry f_6, so that $f = f_6 \circ f_5 \circ \tilde{f}$ is as required.

Proof of (A). Let $p = (a_1, b_1) \dots (a_k, b_k)$. To keep the notation more transparent, we will treat the concrete case when $p = (1,2)(3,4) \dots (n-1, n)$; the generalization is obvious. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined above Theorem 4.1 with $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$. Let $g = e^{\pi i/2} \cdot h$ be h composed with the rotation by $\pi/2$ around the origin. Then g rotates by $\pi/2$ all $z \in \mathbb{R}^2$ with $\|z\| \geq 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with $\|z\| = 2/\sqrt{n}$, as

$$\frac{\pi}{2} + \varepsilon \log \frac{2}{\sqrt{n}} = \frac{\pi}{2} + \frac{\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{2}{\sqrt{2^{K+2}}} = 0.$$

Below we will consider g written in the Cartesian coordinates. We write $\mathbb{R}^n$ as the ℓ_2-sum of $n/2$ copies of $\mathbb{R}^2$ and define f "coordinate-wise": if $x = \sum_{i=1}^n x_i e_i$ then

$$f(x) = f(x_1, x_2, \dots, x_n) = (g(x_1, x_2), g(x_3, x_4), \dots, g(x_{n-1}, x_n)).$$

Since g is a bi-Lipschitz mapping of $\mathbb{R}^2$ onto itself, f is a bi-Lipschitz mapping of $\mathbb{R}^n$ onto itself and the Lipschitz constants are the same, that is, f is a $(\frac{\pi}{\log 2} \cdot \frac{1}{K})$-quasi-isometry. Since g preserves the norm and $g(z) = -g(-z)$ for $z \in \mathbb{R}^2$, f is also norm-preserving and $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$. The projection of e_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is either e_j itself (if $j \in \{l, l+1\}$), or zero. As g rotates by $\pi/2$ on the unit circle and $g(0) = 0$, we have $f(e_{2k-1}) = e_{2k}$ and $f(e_{2k}) = -e_{2k-1}$ for $k \in \{1, \dots, \frac{n}{2}\}$. The projection

$p_l(u_j)$ of u_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is $p_l(u_j) = \frac{1}{\sqrt{n}}(\varepsilon_{l,j}e_l + \varepsilon_{l+1,j}e_{l+1})$, hence $\|p_l(u_j)\| = \frac{2}{\sqrt{n}}$. It follows that $g(p_l(u_j)) = p_l(u_j)$ and $f(u_j) = u_j$ for $j \in \{1, \dots n\}$.

Proof of (B). Again, to keep the notation more transparent, we will treat a concrete case: assume that $\alpha_1 = \dots = \alpha_n = -1$. The generalization is obvious. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined above Theorem 4.1 with $\varepsilon = \frac{2\pi}{\log 2} \cdot \frac{1}{K}$. Let $g = e^{\pi i} \cdot h$ be h composed with the rotation by π around the origin. Then g rotates by π all $z \in \mathbb{R}^2$ with $\|z\| \geq 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with $\|z\| = 2/\sqrt{n}$, as

$$\pi + \varepsilon \log \frac{2}{\sqrt{n}} = \pi + \frac{2\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{2}{\sqrt{2^{K+2}}} = 0.$$

Below we will consider g written in the Cartesian coordinates. We write $\mathbb{R}^n$ as the ℓ_2-sum of $n/2$ copies of $\mathbb{R}^2$ and define f "coordinate-wise": if $x = \sum_{i=1}^n x_i e_i$ then

$$f(x) = f(x_1, x_2, \dots, x_n) = (g(x_1, x_2), g(x_3, x_4), \dots, g(x_{n-1}, x_n)).$$

As in the proof of (A), f is a norm preserving $(\frac{2\pi}{\log 2} \cdot \frac{1}{K})$-quasi-isometry of $\mathbb{R}^n$ onto itself, and $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$. The projection of e_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is either e_j itself (if $j \in \{l, l+1\}$), or zero. As g rotates by π on the unit circle and $g(0) = 0$, we have $f(e_j) = -e_j$ for $j \in \{1, \dots, n\}$. Exactly as in the proof of (A) we get that $f(u_j) = u_j$ for $j \in \{1, \dots n\}$. □

If f is the ε-quasi-isometry from Theorem 4.4 for which $f = -\mathrm{Id}$ on the orthonormal basis $\{e_1, \dots, e_n\}$ and $f = \mathrm{Id}$ on the orthonormal basis $\{u_1, \dots, u_n\}$ (we treated this particular case in the proof of the statement (B)), then $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq 1$ for any linear $T : \mathbb{R}^n \to \mathbb{R}^n$. Indeed, suppose that for some linear $T : \mathbb{R}^n \to \mathbb{R}^n$ we have $\|T(x) - f(x)\| < 1$ for each $x \in B_{\mathbb{R}^n}$. Then

$$\langle T(e_i), e_i \rangle = \langle T(e_i) - f(e_i), e_i \rangle + \langle f(e_i), e_i \rangle \leq -1 + \|T(e_i) - f(e_i)\| < 0.$$

Similarly,

$$\langle T(u_i), u_i \rangle = \langle T(u_i) - f(u_i), u_i \rangle + \langle f(u_i), u_i \rangle \geq 1 - \|T(u_i) - f(u_i)\| > 0.$$

Let A be the matrix of T with respect to the basis $\{e_1, \dots, e_n\}$, and B be the matrix of T with respect to the basis $\{u_1, \dots, u_n\}$. Then $\operatorname{trace} A = \operatorname{trace} B$. At the same time

$$\operatorname{trace} A = \sum_{i=1}^n \langle T(e_i), e_i \rangle < 0,$$

and

$$\operatorname{trace} B = \sum_{i=1}^{n} \langle T(u_i), u_i \rangle > 0,$$

which is a contradiction. As the mapping f in the proof of (B) satisfies moreover $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, it holds also that $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq 1$ for any affine $T : \mathbb{R}^n \to \mathbb{R}^n$.

Acknowledgment.

I would like to thank Joram Lindenstrauss and Piotr Mankiewicz for stimulating discussions on the subject of this paper.

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