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A Note on Nonlinear Approximation with Schauder Bases

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We study the approximation classes $\mathcal{A}_{\alpha,s}$ associated with nonlinear m-term approximation by elements from a quasi-normed Schauder basis in a separable Banach space. We show that there always is a two-sided embedding

$$\mathcal{K}_{\tau_l,s} \hookrightarrow \mathcal{A}_{\alpha,s} \hookrightarrow \mathcal{K}_{\tau_u,s},$$

where $\mathcal{K}_{\tau,s}$ denotes the associated smoothness space. We provide estimates of τ_l and τ_u in terms of quantitative properties of the Schauder basis. The estimates are sharp for so-called quasi-greedy bases. The two-sided embedding can be considered a generalization of the characterization of the approximation class associated with an orthonormal basis $\mathcal{B}$ for a Hilbert space $\mathcal{H}$ where it is well known that

$$\mathcal{A}_{\alpha,s}(\mathcal{B}) = \mathcal{K}_{\tau,s}(\mathcal{B}),$$

with $\alpha = \frac{1}{\tau} - \frac{1}{2}$ and $s \in (0,\infty]$.

Key Words: nonlinear approximation, best m-term approximation, smoothness space, approximation class, Schauder basis, quasi-greedy basis.

1. INTRODUCTION

Let X be a separable Banach space, and let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a quasi-normed Schauder basis of X, i.e. a basis that satisfies $\inf_k \|g_k\|_X > 0$ and $\sup_k \|g_k\|_X < \infty$. For any given $f \in X$, the error associated to the *best m-term* approximation to f from $\mathcal{S}$ is given by

$$\sigma_m(f,\mathcal{S}) = \inf_{\Lambda\subset\mathbb{N}:|\Lambda|=m,\{c_k\}_{k\in\Lambda}\subset\mathbb{C}} \Big\| f - \sum_{k\in\Lambda} c_k g_k \Big\|_X. \qquad (1)$$

We are interested in the characterization of *approximation classes*:

$$\mathcal{A}_{\alpha,s}(\mathcal{S}) = \Big\{ f \in X, \|\{\sigma_m(f,\mathcal{S})\}_{m\geq 1}\|_{\ell_{1/\alpha,s}(\mathbb{N})} < \infty \Big\} \qquad (2)$$

which are defined using the Lorentz (quasi-)norm; for $0 < \tau < \infty$ and $0 < s \leq \infty$:

$$\|(a_m)_{m=1}^{\infty}\|_{\ell_{\tau,s}(\mathbb{N})} := \begin{cases} \left(\sum_{m=1}^{\infty}[m^{1/\tau}a_m^*]^s 1/m\right)^{1/s}, & 0 < s < \infty \\ \sup_{n\in\mathbb{N}} n^{1/\tau}a_n^*, & s = \infty, \end{cases} \quad (3)$$

where $\{a_k^*\}_k$ denotes the decreasing rearrangement of $\{a_k\}_k$.

Remark 1. 1.

1. Notice that $\|\cdot\|_{\ell_{\tau,\tau}} = \|\cdot\|_{\ell_\tau}$.
2. Throughout this paper we will use the notation $V \hookrightarrow W$, where V and W are (quasi-)normed spaces, whenever $V \subset W$ and $\|\cdot\|_W \leq C\|\cdot\|_V$ for some $C < \infty$. It can be verified [DL93] that the Lorentz spaces $\ell_{\tau,s}(\mathbb{N})$, defined by

$$\ell_{\tau,s}(\mathbb{N}) = \{\{c_k\} : \|\{c_k\}\|_{\ell_{\tau,s}} < \infty\},$$

satisfy the continuous embedding $\ell_{\tau_1,s_1}(\mathbb{N}) \hookrightarrow \ell_{\tau_2,s_2}(\mathbb{N})$ provided that $\tau_1 < \tau_2$ or $\tau_2 = \tau_1$ with $s_1 \leq s_2$.

$\mathcal{A}_{\alpha,s}(\mathcal{S})$ is thus basically the set of functions f that can be approximated at a given rate $\mathcal{O}(m^{-\alpha})$ $(0 < \alpha < \infty)$ by m-elements from the Schauder basis. The parameter $0 < s \leq \infty$ is auxiliary and gives a finer classification of the approximation rate.

We also define

$$\|f\|_{\mathcal{A}_{\alpha,s}(\mathcal{S})} := \|f\| + \|\{\sigma_m(f,\mathcal{S})\}_{m=1}^{\infty}\|_{\ell_{1/\alpha,s}(\mathbb{N})} \quad (4)$$

Approximation classes are often related to smoothness classes. For $\tau \in (0,\infty)$ and $s \in (0,\infty]$, we let $\mathcal{K}_{\tau,s}(\mathcal{S},M)$ denote the set

$$\operatorname{clos}_X\left\{f \in X \mid \exists\Lambda \subset \mathbb{N}, |\Lambda| < \infty,\, f = \sum_{k\in\Lambda} c_k g_k,\, \|\{c_k\}\|_{\ell_{\tau,s}(\mathbb{N})} \leq M\right\}.$$

Then we define

$$\mathcal{K}_{\tau,s}(\mathcal{S}) := \cup_{M>0}\mathcal{K}_{\tau,s}(\mathcal{S},M), \quad (5)$$

with

$$\|f\|_{\mathcal{K}_{\tau,s}(\mathcal{S})} = \inf\{M : f \in \mathcal{K}_{\tau,s}(\mathcal{S},M)\}.$$

Remark 1. 2. In a Hilbert space $\mathcal{H}$, consider $\mathcal{K}_{\tau,s}(\mathcal{S})$ with $\tau \in (0,2)$ and suppose that the Schauder basis $\mathcal{S}$ is hilbertian, i.e. for every ℓ_2 sequence of scalars $\{c_k\}$, the sum $\sum_k c_k g_k$ is convergent in $\mathcal{H}$. As $\ell_{\tau,s}(\mathbb{N}) \hookrightarrow \ell_2(\mathbb{N})$ one can check that

$$\mathcal{K}_{\tau,s}(\mathcal{S}) = \left\{ f \in X, f = \sum_k c_k g_k, \|(c_k)\|_{\ell_{\tau,s}(\mathbb{N})} < \infty \right\}, \tag{6}$$

where Fatou's Lemma can be used to obtain the $\subseteq$ inclusion in (6).

The following characterization was proved by Stechkin [Ste55] for the case $\tau = 1$ and for general τ by DeVore and Temlyakov [DT96] when the Schauder basis $\mathcal{S}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$.

THEOREM 1.1 ([Ste55, DT96]). *If $\mathcal{B} = \{h_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\mathcal{A}_{\alpha,s}(\mathcal{B}) = \mathcal{K}_{\tau,s}(\mathcal{B}) \tag{7}$$

with $\alpha = \frac{1}{\tau} - \frac{1}{2}$. Moreover, the norm is given by

$$\|f\|_{\mathcal{K}_{\tau,s}(\mathcal{B})} = \|\{\langle f, h_k\rangle\}_{k\in\mathbb{N}}\|_{\ell_{\tau,s}(\mathbb{N})} \asymp \|f\|_{\mathcal{A}_{\alpha,s}(\mathcal{B})}. \tag{8}$$

The fundamental tools to prove these results are Hardy's inequalities. Less is known when $\mathcal{S}$ is not an orthonormal basis and the purpose of this note is to generalize Theorem 1.1 to general Schauder bases for a Banach space. This will be done in Section 2. In Section 3 we will show that for so-called quasi-greedy bases the results of Section 2 are the best possible. We consider some specific examples of the main result for Banach spaces which are uniformly smooth and uniformly convex in Section 4.

2. MAIN RESULTS

Let us consider a quasi-normed Schauder basis $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ for a Banach space X. Since the basis is quasi-normed it is known, see [You80], that there exist constants $0 < A \le B < \infty$ such that for every $f = \sum_k c_k(f) g_k \in X$ we have

$$A\|\{c_k(f)\}\|_{\ell_\infty} \le \|f\|_X \le B\|\{c_k(f)\}\|_{\ell_1}.$$

For any pair $1 \le p \le q \le \infty$ we can thus ask whether there are constants $0 < A_q \le A_p < \infty$ for which the estimate

$$A_q\|\{c_k(f)\}\|_{\ell_q} \le \|f\|_X \le B_p\|\{c_k(f)\}\|_{\ell_p} \tag{9}$$

holds for every $f \in X$. For Schauder bases we have the following result, generalizing Theorem 1.1:

THEOREM 2.1. *Let $\mathcal{S}$ be a quasi-normed Schauder basis for a Banach space X. For every pair (p,q), $1 \le p \le q \le \infty$ such that (9) is satisfied, we have for $s \in (0,\infty]$,*

$$\mathcal{K}_{\tau_l,s}(\mathcal{S}) \hookrightarrow \mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow \mathcal{K}_{\tau_u,s}(\mathcal{S}), \tag{10}$$

with

$$\frac{1}{\tau_l} = \frac{1}{p} + \alpha \quad \text{and} \quad \frac{1}{\tau_u} = \frac{1}{q} + \alpha.$$

Remark 2. 1.

1. For a general Schauder basis we get a “weaker” result than for an orthonormal basis in the sense that the approximation class is not entirely characterized as a smoothness space by Theorem 2.1. Indeed, the only case where the Theorem gives an exact characterization is when $p = q$ can be realized in (9), that is when X is isometric to ℓ_p. The Theorem then reduces to a variant of Hardy's inequality. But it is only natural that we have to pay a price to use a less structured basis.

2. That we need some structure (and not just a set with dense span) to get a result like Theorem 2.1 will be demonstrated at the end of Section 4 with an explicit example.

We now give the proof of Theorem 2.1. We will use some basic properties of the real interpolation method of Lyons and Peetre. The reader can find more information on this topic and the notation used below in [DL93, Chap. 6].

Proof of Theorem 2.1. Let $1 \le p \le q \le \infty$ be such that (9) is satisfied. Given $\alpha \in (0,\infty)$, we take τ with $\tau < p$ such that $\alpha < \tau^{-1} - p^{-1}$, this choice will be justified later. We put $Y = \mathcal{K}_{\tau,\tau}(\mathcal{S})$. The proof has two steps; First, we will prove a two-sided embedding of the approximation class in interpolation spaces of the type $(X,Y)_{\beta,s}$. Then we will find two-sided embeddings of the interpolation spaces $(X,Y)_{\beta,s}$ into spaces that can be identified with sequence spaces.

For $S = \sum_{k=1}^{n} c_k g_{n_k}$, using Hölder's inequality and then (9),

$$\begin{aligned}\|S\|_Y &\le \left(\sum_{k=1}^{n} |c_k|^\tau\right)^{1/\tau} \\ &\le n^{1/\tau - 1/q}\left(\sum_{k=1}^{n} |c_k|^q\right)^{1/q} \\ &\le \frac{1}{A} n^{1/\tau-1/q}\|S\|_X.\end{aligned}$$

Hence, we have the Bernstein inequality with exponent $r_1 := \frac{1}{\tau} - \frac{1}{q} > 0$. Notice that $0 < \alpha < \tau^{-1} - p^{-1} \le r_1$. It follows that for $s \in (0, \infty]$, see [DL93, Chap. 7],

$$\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow (X, Y)_{\alpha/r_1, s}. \tag{11}$$

To get a Jackson type inequality we will use the following notation; for a given sequence $c = \{c_k\}$ we let $c^{p/2}$ denote the sequence $\{|c_k|^{p/2}\}_{k\in\mathbb{N}}$ and we let $\Theta_m(c)$ be the thresholding operator that keeps only the m largest elements of c. For $f = \sum_k c_k g_k \in Y$ we let

$$f_m = \sum_{k\in\mathbb{N}} [\Theta_m(c)]_k g_k.$$

We first notice that by (9), we have

$$\|f\|_X \le B\|c^{p/2}\|_{\ell_2(\mathbb{N})}^{2/p}.$$

Moreover,

$$\begin{aligned}\sigma_m(f) &\le \|f - f_m\|_X \\ &\le B\|c - \Theta_m(c)\|_{\ell_p(\mathbb{N})} \\ &= B\|c^{p/2} - \Theta_m(c^{p/2})\|_{\ell_2(\mathbb{N})}^{2/p} \\ &= B[\sigma_m(c^{p/2})]^{2/p}.\end{aligned} \tag{12}$$

We can use Theorem 1.1 to estimate $\sigma_m(c^{p/2})$ since $c^{p/2} \in \ell^{2\tau/p}(\mathbb{N}) \hookrightarrow \ell_{2\tau/p,\infty}(\mathbb{N})$, with $\|c^{p/2}\|_{\ell_{2\tau/p,\infty}(\mathbb{N})} = \|c\|_{\ell_{\tau,\infty}(\mathbb{N})}^{p/2}$. We let $\gamma := \frac{p}{2\tau} - \frac{1}{2}$ and from Theorem 1.1 we have

$$\sigma_m(c^{p/2}) \le Cm^{-\gamma}\|c^{p/2}\|_{\ell_{2\tau/p,\infty}(\mathbb{N})} = Cm^{-\gamma}\|c\|_{\ell_{\tau,\infty}(\mathbb{N})}^{p/2} \le \tilde{C}m^{-\gamma}\|c^{p/2}\|_{\ell_{\tau,\tau}(\mathbb{N})}^{p/2}.$$

Taking into account the exponent $2/p$ in (12), we introduce the constant r_2 defined by $r_2 := \frac{2}{p}\gamma = \frac{1}{\tau} - \frac{1}{p}$, and obtain the Jackson inequality

$$\sigma_m(f) \leq Cm^{-r_2}\|f\|_Y. \tag{13}$$

The choice of τ is such that $0 < \alpha < r_2$, and for $s \in (0,\infty]$ we have [DL93, Chap. 7],

$$(X,Y)_{\alpha/r_2,s} \hookrightarrow \mathcal{A}_{\alpha,s}(\mathcal{S}). \tag{14}$$

Now, we will look closer at the spaces $(X,Y)_{\theta,s}$ for $\theta \in (0,1)$. Define the operator T by

$$T\left(\sum_{k=1}^{\infty} c_k g_k\right) = \{c_k\}_{k=1}^{\infty}.$$

Notice that T is continuous as a mapping on the following spaces

$$T : X \to \ell_{q,q}(\mathbb{N}),$$
$$T : Y \to \ell_{\tau,\tau}(\mathbb{N}).$$

Hence, by interpolation, for $\theta \in (0,1)$ and $s \in (0,\infty]$,

$$T : (X,Y)_{\theta,s} \to (\ell_{q,q}(\mathbb{N}), \ell_{\tau,\tau}(\mathbb{N}))_{\theta,s}$$

is continuous. Conversely, we define (formally)

$$U(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k g_k,$$

and we see that U is continuous as a mapping on:

$$U : \ell_{p,p}(\mathbb{N}) \to X,$$
$$U : \ell_{\tau,\tau}(\mathbb{N}) \to Y.$$

Thus, for $\theta \in (0,1)$ and $s \in (0,\infty]$,

$$U : (\ell_{p,p}(\mathbb{N}), \ell_{\tau,\tau}(\mathbb{N}))_{\theta,s} \to (X,Y)_{\theta,s}$$

is continuous. Combining this with (11), (14), and using the characterization of the interpolation classes between ℓ_p spaces, see [DeV98, p. 39], we finally obtain

$$\mathcal{K}_{\tau_l,s}(\mathcal{S}) = U\ell_{\tau_l,s}(\mathbb{N}) = U(\ell_{p,p}(\mathbb{N}), \ell_{\tau,\tau}(\mathbb{N}))_{\alpha/r_2,s} \hookrightarrow (X,Y)_{\alpha/r_2,s} \hookrightarrow \mathcal{A}_{\alpha,s}(\mathcal{S}),$$

and

$$T\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow T(X,Y)_{\alpha/r_1,s} \hookrightarrow (\ell_{q,q}(\mathbb{N}), \ell_{\tau,\tau}(\mathbb{N}))_{\alpha/r_1,s} = \ell_{\tau_u,s}(\mathbb{N}), \tag{15}$$

with

$$\frac{1}{\tau_l} := \left(1 - \frac{\alpha}{r_2}\right)\frac{1}{p} + \frac{\alpha}{r_2}\frac{1}{\tau} \quad \text{and} \quad \frac{1}{\tau_u} := \left(1 - \frac{\alpha}{r_1}\right)\frac{1}{q} + \frac{\alpha}{r_1}\frac{1}{\tau},$$

which can be reduced to

$$\frac{1}{\tau_l} = \frac{1}{p} + \alpha \quad \text{and} \quad \frac{1}{\tau_u} = \frac{1}{q} + \alpha.$$

Notice that since $\mathcal{S}$ is a Schauder basis, (15) implies that

$$\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow \mathcal{K}_{\tau_u,s}(\mathcal{S}),$$

which completes the proof. □

3. SHARPNESS RESULTS

We now consider the sharpness of Theorem 2.1. First we have to specify what we mean by a sharp result of this type. Given a Schauder basis $\mathcal{S}$ for a Banach space X it makes sense to define the following quantities:

$$\lambda(\mathcal{S}) := \inf\{q : \text{lower bound of (9) holds for some } A_q > 0\} \tag{16}$$

$$\mu(\mathcal{S}) := \sup\{p : \text{upper bound of (9) holds for some } B_p < \infty\}, \tag{17}$$

and we clearly always have $1 \leq \mu(\mathcal{S}) \leq \lambda(\mathcal{S}) \leq \infty$. For uniformly smooth and uniformly convex Banach spaces we have better estimates on $\lambda(\mathcal{S})$ and $\mu(\mathcal{S})$, this will be discussed in Section 4.

Theorem 2.1 says that for any $p \leq \mu(\mathcal{S})$ and $q \geq \lambda(\mathcal{S})$ for which (9) holds we have the embedding lines given by $1/\tau_l = 1/p + \alpha$ and $1/\tau_u = 1/q + \alpha$. The sharpness of these embedding lines are in the following sense. Suppose that we have the embedding line $1/\tau_l = 1/\tilde{p}+\alpha$, then Proposition 3.1 below will show that $\tilde{p} \leq \mu(\mathcal{S})$. If we in addition assume that the basis has the so-called quasi-greedy property and we are given the upper embedding line $1/\tau_u = 1/\tilde{q} + \alpha$, then we show in Proposition 3.2 that $\tilde{q} \geq \lambda(\mathcal{S})$.

First we consider the lower embedding.

PROPOSITION 3.1. *Let $\mathcal{S}$ be a Schauder basis for X and suppose that $\tilde{p} > 1$ is such that $\mathcal{K}_{\tau,s}(\mathcal{S}) \hookrightarrow \mathcal{A}_{\alpha,s}(\mathcal{S})$ for every $\alpha > 0$ and $\tau := (\alpha+1/\tilde{p})^{-1}$. Then $\tilde{p} \leq \mu(\mathcal{S})$, where $\mu(\mathcal{S})$ is defined in (17).*

Proof. We notice that $\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow X$ for all $\alpha > 0$. Let $1 < \tau < \tilde{p}$. By taking $\alpha = 1/\tau - 1/\tilde{p}$, we deduce from the embedding

$$\mathcal{K}_{\tau,\tau}(\mathcal{S}) \hookrightarrow \mathcal{A}_{\alpha,\tau}(\mathcal{S}) \hookrightarrow X$$

that $\mathcal{K}_{\tau,\tau}(\mathcal{S}) \hookrightarrow X$, that is to say $\|f\|_X \leq B_\tau \|\{c_k(f)\}\|_{\ell_\tau}$. As this is true for any $1 < \tau < \tilde{p}$, $\mu(\mathcal{S}) \geq \tilde{p}$. □

Next we consider the upper embedding in Theorem 2.1, but first we need a definition and a technical Lemma. Sharpness in the upper embedding for quasi-greedy bases will be proved in Proposition 3.2.

Definition 3.1. Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a quasi-normed Schauder basis for the Banach space X. We call $\mathcal{S}$ a quasi-greedy basis if for each $f = \sum_k c_k g_k \in X$, and $\{c_{\phi(k)}\}$ the decreasing rearrangement of $\{c_k\}$, we have $\left\| \sum_{k=1}^N c_{\phi(k)} g_{\phi(k)} - f \right\|_X \to 0$ as $N \to \infty$.

Remark 3. 1. It is clear that every quasi-normed unconditional basis for X will also be quasi-greedy but it is known that the converse result is false [Woj00], so being quasi-greedy is a weaker condition than being unconditional.

The following Lemma was proved in the special case $p = 2$ in [Woj00], and the authors would like to thank Denka Kutzarova-Ford and Stephen Dilworth for pointing out to us that the technique used in [Woj00] also works in the more general setting presented below.

Lemma 3.1. *Let $\mathcal{S} = \{g_k\}$ be a quasi-greedy basis for X and suppose that there is a constant c such that for any finite subset $A \subset \mathbb{N}$,*

$$\left\| \sum_{k\in A} \pm g_k \right\|_X \geq c|A|^{1/q}.$$

Then, for each $f = \sum_{k\in\mathbb{N}} c_k g_k \in X$,

$$\|\{c_k\}\|_{\ell_{q,\infty}(\mathbb{N})} \leq C\|f\|_X.$$

Proof. Let $f = \sum_{k\in\mathbb{N}} c_k g_k \in X$, and let $\{c_{\phi(k)}\}$ be a decreasing rearrangement of $\{c_k\}$, i.e. a rearrangement for which $|c_{\phi(1)}| \geq |c_{\phi(2)}| \geq \cdots$. Since $\mathcal{S}$ is quasi-greedy there is a constant C depending only on $\mathcal{S}$ [Woj00, Theorem 1] such that $\sup_N \| \sum_{k=1}^N c_{\phi(k)} g_{\phi(k)} \|_X \leq C\|f\|_X$. Using the Abel transform we get for any increasing sequence $\{\alpha_k\}$ of positive numbers that

$\sup_N \| \sum_{k=1}^N \alpha_k c_{\phi(k)} g_{\phi(k)} \|_X \leq C(\sup_k \alpha_k) \|f\|_X$. Thus, for every $N \geq 1$ for which $c_{\phi(N)} \neq 0$ and $\alpha_k = |c_{\phi(N)}||c_{\phi(k)}|^{-1}$, $k = 1, 2, \ldots, N$, we have

$$|c_{\phi(N)}| N^{1/q} \leq c^{-1} \left\| \sum_{k=1}^N \frac{c_{\phi(k)} |c_{\phi(N)}|}{|c_{\phi(k)}|} g_{\phi(k)} \right\|_X \leq c^{-1} C \|f\|_X.$$

It follows at once that $\|\{c_k\}\|_{\ell_{q,\infty}(\mathbb{N})} \leq c^{-1} C \|f\|_X$. □

We now turn to the sharpness result for the upper embedding for quasi-greedy bases.

Proposition 3.2. *Let $\mathcal{S} = \{g_k\}_{k \in \mathbb{N}}$ be a quasi-greedy basis for X and suppose $1 < \tilde{q} < \infty$ is such that for every $\alpha > 0$, we have $\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow \mathcal{K}_{\tau,s}(\mathcal{S})$ for $1/\tau = \alpha + 1/\tilde{q}$. Then $\lambda(\mathcal{S}) \leq \tilde{q}$, where $\lambda(\mathcal{S})$ is defined by (16).*

Proof. Let $\alpha > 0$, and let $A \subset \mathbb{N}$ with $|A| = n$. Take $\varepsilon \in \{-1, 1\}^{\mathbb{N}}$, and put $\psi = \sum_{k \in A} \varepsilon_k g_k$. Then $\|\psi\|_{\mathcal{K}_{\tau,s}} = n^{1/\tau}$, and by using $\mathcal{A}_{\alpha,s}(\mathcal{S}) \hookrightarrow \mathcal{K}_{\tau,s}(\mathcal{S})$ together with [DL93, Chap. 7; Theorem 9.3] we obtain

$$n^{1/\tau} = \|\psi\|_{\mathcal{K}_{\tau,s}(\mathcal{S})} \leq C \|\psi\|_{\mathcal{A}_{\alpha,s}(\mathcal{S})} \leq \tilde{C} n^{\alpha} \|\psi\|_X.$$

From this we deduce that for $A \subset \mathbb{N}$, $|A| = n$,

$$\left\| \sum_{k \in A} \pm g_k \right\|_X \geq \tilde{C}^{-1} n^{1/\tilde{q}}.$$

Hence, from Lemma 3.1 we conclude that $\|\{c_k(f)\}\|_{\ell_{\tilde{q},\infty}(\mathbb{N})} \leq C \|f\|_X$ which clearly implies that $\lambda(\mathcal{S}) \leq \tilde{q}$. □

4. EXAMPLES

We will now present some examples of the use of Theorem 2.1 for some specific Schauder bases and for general Schauder bases in Banach spaces with additional structure. First we state an easy corollary of Theorem 2.1 that generalizes Theorem 1.1 to Riesz bases:

Corollary 4.1. *Suppose $\mathcal{S}$ is a Riesz basis for a Hilbert space $\mathcal{H}$, i.e. that (9) holds with $p = q = 2$, then for $0 < \tau < 2$ and $s \in (0, \infty]$,*

$$\mathcal{A}_{\alpha,s}(\mathcal{S}) = \mathcal{K}_{\tau,s}(\mathcal{S}),$$

with $\alpha = \frac{1}{\tau} - \frac{1}{2}$.

Next we consider Schauder bases in uniformly convex and uniformly smooth Banach spaces. The following fundamental result is known about Schauder bases in such Banach spaces:

THEOREM 4.1 ([GG71]). *Let $\mathcal{S}$ be a quasi-normed Schauder basis for a Banach space X which is both uniformly smooth and uniformly convex. Then $1 < \mu(\mathcal{S}) \leq \lambda(\mathcal{S}) < \infty$.*

This Theorem shows that whenever the Banach space X is uniformly convex and uniformly smooth we are guaranteed to get better embedding lines from Theorem 2.1 than the ones for the "worst case" scenario where $\mu(\mathcal{S}) = 1$ and $\lambda(\mathcal{S}) = \infty$. How much improvement we get in uniformly smooth and uniformly convex Banach spaces clearly depends on the specific structure of the Basis $\mathcal{S}$. In fact, any pair of p and q with $1 < p \leq 2 \leq q < \infty$ is realized by some Schauder basis in some Banach space, as the following Theorem shows.

THEOREM 4.2 ([GG71]). *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Given a pair of numbers p and q satisfying $1 < p \leq 2 \leq q < \infty$, there exists a Schauder basis $\mathcal{S}$ for $\mathcal{H}$ with the property that $\mu(\mathcal{S}) = p$ and $\lambda(\mathcal{S}) = q$.*

Remark 4. 1. It depends on the properties of the basis $\mathcal{S}$ whether $\lambda(\mathcal{S})$ and $\mu(\mathcal{S})$ are actually attained or not. For the canonical basis $\mathcal{C}$ in ℓ_τ, $1 < \tau < \infty$, $\lambda(\mathcal{C}) = \tau = \mu(\mathcal{C})$ and (9) obviously holds for $p = q = \tau$ (in fact, it is clear that X is isometric to ℓ_τ in all cases where $p = q = \tau$ is realized).

For the Lorentz space $\ell_{\tau,s}$ we also have $\lambda(\mathcal{C}) = \tau = \mu(\mathcal{C})$. But for $s < \tau$, the upper bound in (9) fails for $p = \tau$, and the lower bound in (9) fails for $q = \tau$ when $s > \tau$.

We conclude this paper by considering collections of normalized vectors that do not form Schauder bases. So far we have only considered the relationship between the approximation and smoothness spaces associated with Schauder bases. We can define the approximation classes and smoothness spaces by the analogs of (2) and (5), respectively, for any set $\mathcal{U}$ of unit vectors with dense span in X. One can then pose the question whether it possible to get results like Theorem 2.1 for more general sets with dense span. This is not the case, in general, and we conclude this paper by giving an example of a spanning non-redundant set $\mathcal{U}$ in a Hilbert space $\mathcal{H}$, which fails to be a Schauder basis for $\mathcal{H}$ and for which the upper embedding of

Theorem 2.1 fails to be true no matter which combination of parameters α and τ' one chooses.

PROPOSITION 4.1. *Let $\mathcal{H} = \bigoplus_{j\geq 0} \mathcal{V}_j$ be an orthogonal decomposition of $\mathcal{H}$ into two-dimensional subspaces. Let $\{e_{2j}, e_{2j+1}\}$ be a normalized basis of $\mathcal{V}_j$ such that $\langle e_{2j}, e_{2j+1}\rangle = \cos\phi_j$, $\phi_j > 0$, and $\phi_j \to 0$. Let $\mathcal{U} = \{e_k\}_{k\geq 0}$. Then*

$$\mathcal{A}_{\alpha,s}(\mathcal{U}) \not\hookrightarrow \mathcal{K}_{\tau',s'}(\mathcal{U})$$

for any combination of parameters $0 < \alpha, \tau' < \infty$, and $0 < s, s' \leq \infty$.

Proof: We define a sequence $\{f_j\}$ for which

$$\|f_j\|_{\mathcal{A}_{\alpha,s}(\mathcal{U})} \to 0 \quad \text{and} \quad \|f_j\|_{\mathcal{K}_{\tau',s'}(\mathcal{U})} \geq 1,$$

which will prevent any type of continuous embedding of the approximation class into the smoothness space. More precisely, we let

$$f_j = \cos\phi_j e_{2j} - e_{2j+1}$$

and check that $\|f_j\| = |\sin\phi_j| \to 0$. Hence it is clear that

$$\|f_j\|_{\mathcal{A}_{\alpha,s}(\mathcal{U})} \leq 3\,\|f_j\| \to 0$$

while

$$\|f_j\|_{\mathcal{K}_{\tau',s'}(\mathcal{U})} \geq \|\{\ldots, \cos\phi_j, 1, \ldots\}\|_{\ell_\infty} \geq 1.$$

□

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