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Let

$$\mathbb{B}^2 := \left\{(x, y) \in R^2 : \, x^2 + y^2 \leq 1\right\} \tag{1}$$

denote the unit disc on the plane and

$$u_m(t) := \frac{1}{\sqrt{\pi}} \frac{\sin{(m+1)} \arccos t}{\sqrt{1-t^2}}, \tag{2}$$

$m = 0, 1, \ldots, t \in [-1, 1]$, are the Chebyshev polynomials of the second kind. For an arbitrary sequence of real phases $\{\varphi_m\}_{m=0}^{\infty}$, we get on $\mathbb{B}^2$ the corresponding discrete sequence of Chebyshev ridge polynomials

$$\left\{\left\{u_m\left(x\cos\left(\frac{k\pi}{m+1}+\varphi_m\right)+y\sin\left(\frac{k\pi}{m+1}+\varphi_m\right)\right)\right\}_{k=0}^{m}\right\}_{m=0}^{\infty}. \tag{3}$$

These systems are very useful tool in the theory of approximation of functions by feed–forward neural networks [1], [2]. It is known [2] that for an arbitrary sequence of real phases $\{\varphi_m\}_{m=0}^{\infty}$, the system (3) is a complete orthonormal system in $L^2(\mathbb{B}^2)$. We consider convergence problem to zero for Fourier coefficients $(0 \leq k < m+1, \; m = 0, 1, \ldots)$

$$a_m(f, \, k, \, \varphi_m) := \int_{\mathbb{B}^2} f(x,y)\, u_m\left(x\cos\left(\frac{k\pi}{m+1}+\varphi_m\right)+y\sin\left(\frac{k\pi}{m+1}+\varphi_m\right)\right) dx\, dy \tag{4}$$

of a function $f \in L^p(\mathbb{B}^2)$ with respect to the systems (3). The partial L^p-integral moduli of continuity of a function $f \in L^p(\mathbb{B}^2)$ are defined as follows

$$\omega_1(\delta; f)_p := \sup_{|h|\leq\delta} \left(\int_{\mathbb{B}^2 \bigcap \mathbb{B}^2(1,h)} |f(x+h,y)-f(x,y)|^p \, dxdy\right)^{\frac{1}{p}}, \tag{5}$$

and

$$\omega_2(\delta; f)_p := \sup_{|h|\le\delta} \left(\int_{\mathbb{B}^2 \bigcap \mathbb{B}^2(2,h)} |f(x, y+h) - f(x,y)|^p \, dxdy \right)^{\frac{1}{p}} . \tag{6}$$

where

$$\mathbb{B}^2(1,h) := \left\{ (x,y) \in R^2 : \ (x+h, y) \in \mathbb{B}^2 \right\}, \quad \mathbb{B}^2(2,h) := \left\{ (x,y) \in R^2 : \ (x, y+h) \in \mathbb{B}^2 \right\}. \tag{7}$$

In the present article we shall prove the following theorems.

Theorem 1 *Let $\{\varphi_m\}_{m=0}^{\infty}$ be an arbitrary sequence of real numbers and $f \in L^p(\mathbb{B}^2)$, $p > \frac{3}{2}$. Then the ridge Chebyshev –Fourier coefficients of f tend to zero:*

$$\lim_{m\to\infty} \max_{0\le k\le m} |a_m(f,\, k,\, \varphi_m)| = 0. \tag{8}$$

Theorem 2 *There exists a function $g \in L^{\frac{3}{2}}(\mathbb{B}^2)$ such that*

$$\omega_1(\delta; g)_{\frac{3}{2}} = O\left(\left(\frac{1}{\lg \frac{1}{\delta}} \right)^{\frac{1}{3}} \right), \ (\delta \to 0+); \quad \omega_2(\delta; g)_{\frac{3}{2}} = 0, \ (\delta \in (0,1)) \tag{9}$$

and for each sequence $\{\varphi_m\}_{m=0}^{\infty}$ the following inequality holds true

$$\limsup_{m\to\infty} \max_{0\le k\le m} |a_m(g, k, \varphi_m)| \ge C_1 > 0, \tag{10}$$

where C_1 is an absolute constant.

The next statement follows from Theorem 2.

Corollary 1 *There exists a function $g \in L^{\frac{3}{2}}(\mathbb{B}^2)$ that satisfies (9) and for each sequence $\{\varphi_m\}_{m=0}^{\infty}$ Fourier series of g with respect to the system (3) diverges in $L^{\frac{3}{2}}(\mathbb{B}^2)$.*

Proof of the Corollary. First we prove that for $m = 0, 1, \ldots,$ $k = 0, 1, \ldots\ m$, and for each sequence $\{\varphi_m\}_{m=0}^{\infty}$ we have

$$\int_{\mathbb{B}^2} \left| u_m \left(x \cos\left(\frac{k\pi}{m+1} + \varphi_m \right) + y \sin\left(\frac{k\pi}{m+1} + \varphi_m \right) \right) \right| dxdy \ \ge \ \frac{\sqrt{\pi}}{2}. \tag{11}$$

Indeed, according to (1) and (2)

$$\int_{\mathbb{B}^2} \left| u_m \left(x \cos \left(\frac{k\pi}{m+1} + \varphi_m \right) + y \sin \left(\frac{k\pi}{m+1} + \varphi_m \right) \right) \right| \, dxdy$$
$$= \int_{\mathbb{B}^2} |u_m(x)| dxdy = \frac{2}{\sqrt{\pi}} \int_{-1}^{1} |\sin{(m+1)} \arccos x| \, dx$$
$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi} |\sin{(m+1)\vartheta}| \sin\vartheta d\vartheta \ \geq \frac{2}{\sqrt{\pi}} \int_0^{\pi} (\sin{(m+1)\vartheta} \ \sin\vartheta)^2 \ d\vartheta$$
$$= \frac{1}{2\sqrt{\pi}} \int_0^{\pi} (1 - \cos 2(m+1)\vartheta)(1 - \cos 2\vartheta) \, d\vartheta = \frac{\sqrt{\pi}}{2} \, .$$

Consequently for the function g from Theorem 2 we get

$$\max_{0 \le k \le m} \left\| a_m(g,\, k,\, \varphi_m) \, u_m \left(x \cos \left(\frac{k\pi}{m+1} + \varphi_m \right) + y \sin \left(\frac{k\pi}{m+1} + \varphi_m \right) \right) \right\|_{\frac{3}{2}}$$
$$\geq C_2 \max_{0 \le k \le m} |a_m(g,\, k,\, \varphi_m)|$$

for each sequence $\{\varphi_m\}_{m=0}^{\infty}$ and $m = 0,\, 1,\, \ldots,$ where C_2 is an absolute positive constant. Now the Corollary follows from (10).

Proof of Theorem 1 . First we note that for each $\epsilon \in (0,1)$ there exists a constant B_ϵ such that

$$\int_{\mathbb{B}^2} |u_m(x)|^{3-\epsilon} dxdy \leq \ B_\epsilon, \quad m = 0,\, 1,\, \ldots \, . \tag{12}$$

Indeed

$$\int_{\mathbb{B}^2} |u_m(x)|^{3-\epsilon} dxdy \ = 2 \left(\frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \int_{-1}^{1} |\sin{(m+1)} \arccos x|^{3-\epsilon} \ \left(\sqrt{1-x^2} \right)^{\epsilon-2} dx$$
$$= 4 \left(\frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \int_0^{\frac{\pi}{2}} |\sin{(m+1)\vartheta}|^{3-\epsilon} \ (\sin\vartheta)^{\epsilon-1} \ d\vartheta$$
$$= 4 \left(\frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \left((m+1)^{3-\epsilon} \ \pi^{1-\epsilon} \, 2^{\epsilon-1} \int_0^{\frac{\pi}{m+1}} \vartheta^2 \, d\vartheta \ + 4 \, \pi^{1-\epsilon} \, 2^{\epsilon-1} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{2}} \vartheta^{\epsilon-1} \, d\vartheta \right)$$
$$= 4 \left(\frac{1}{\sqrt{\pi}} \right)^{3-\epsilon} \left(o(1) \ + \frac{4\pi}{2\epsilon} \right) \quad \text{as} \quad m \to \infty \, .$$

Now let $p > \frac{3}{2}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then using Helder's inequality and (12) we obtain that for an arbitrary f from $L^p(\mathbb{B}^2)$ the following inequality holds true:

$$\max_{0 \le k \le m} |a_m(f, k, \varphi_m)| \le \|f\|_p \, \|u_m\|_q, \quad m = 0, 1, \dots .$$

For a given positive δ we can find a function $h = h_\delta \in L^2(\mathbb{B}^2)$ such that

$$\|f - h\|_p \le \delta.$$

Consequently (cf.(12))

$$\begin{aligned} &\max_{0 \le k \le m} |a_m(f, k, \varphi_m)| \le \|f - h\|_p \, \|u_m\|_q + \max_{0 \le k \le m} |a_m(h, k, \varphi_m)| \\ &\le O_q(\delta) + o(1) \quad \text{as} \quad m \to \infty \end{aligned}$$

Theorem 1 is proved. The next statement is essential in the proof of Theorem 2.

Lemma 1 *For each m, $m \ge m_0$, there exists a function $q_{m-1}(x)$ of one variable, defined on [-1,1] such that the function $Q_{m-1}(x, y)$ defined by*

$$Q_{m-1}(x, y) := q_{m-1}(x) \quad for \quad (x, y) \in \mathbb{B}^2, \tag{13}$$

satisfies the following conditions:

$$\max_{0 \le k \le m-1} |a_{m-1}(Q_{m-1}, k, \varphi)| \ge C_3 \, (\log m)^{\frac{1}{3}} \quad for\, all\, real \quad \varphi, \tag{14}$$

$$\|Q_{m-1}\|_{\frac{3}{2}} \le C_4\,, \tag{15}$$

$$\omega_1(\delta; Q_{m-1})_{\frac{3}{2}} \le \omega_{m-1}(\delta) := \begin{cases} C_5 (m^2\delta)^{\frac{2}{3}} \, (\log m)^{-\frac{2}{3}} & for \quad 0 \le \delta \le \frac{2}{m^2} \\ 2C_5 \quad for \quad \delta > \frac{2}{m^2} \end{cases} \tag{16}$$

$$\omega_2(\delta; Q_{m-1})_{\frac{3}{2}} = 0 \quad for\ all \quad \delta \in (0, 1)\,, \tag{17}$$

where C_3 , C_4 , m_0 , C_5 are positive absolute constants and

$$a_m(f,\, k,\, \varphi) := \int_{\mathbb{B}^2} f(x, y) \, u_m \left(x \cos\left(\frac{k\pi}{m+1} + \varphi \right) + y \sin\left(\frac{k\pi}{m+1} + \varphi \right) \right) \, dx \, dy. \tag{18}$$

Proof of the Lemma. Consider the functions $f_k^{(m)}(x),\ -1 \le x \le 1,\ k = 1, 2, \ldots, [\sqrt{m}]$:

$$f_k^{(m)}(x) = \begin{cases} \frac{1}{k^2} & \text{for } x \in \left[\cos\frac{(2k+1)\pi}{m},\ \cos\frac{2k\pi}{m}\right], \\ 0 & \text{otherwise on } [-1,1] \end{cases} \tag{19}$$

and let

$$q_{m-1}(x) := \frac{m^2}{(\log m)^{\frac{2}{3}}} \sum_{k=1}^{[\sqrt{m}]} f_k^{(m)}(x). \tag{20}$$

Now introduce the function $Q_{m-1}(x,y)$ defined on the unit disc $\mathbb{B}^2$

$$Q_{m-1}(x,y) := q_{m-1}(x) \quad \text{for} \quad (x,y) \in \mathbb{B}^2. \tag{21}$$

First we prove that for $m \ge m_0^{(1)}$

$$\|Q_{m-1}\|_{\frac{3}{2}} \le C_4, \tag{22}$$

for some absolute constant $m_0^{(1)}$. Indeed (cf.(19), (20), (21))

$$\begin{aligned}
&\int_{\mathbb{B}^2} |Q_{m-1}(x,y)|^{\frac{3}{2}} dxdy = 2\int_{-1}^{1} |q_{m-1}(x)|^{\frac{3}{2}} \sqrt{1-x^2}dx \\
&= 2\sum_{l=1}^{[\sqrt{m}]} \int_{\cos\frac{(2l+1)\pi}{m}}^{\cos\frac{2l\pi}{m}} |q_{m-1}(x)|^{\frac{3}{2}} \sqrt{1-x^2}dx = 2\frac{m^3}{\log m} \sum_{l=1}^{[\sqrt{m}]} \frac{1}{l^3} \int_{\cos\frac{(2l+1)\pi}{m}}^{\cos\frac{2l\pi}{m}} \sqrt{1-x^2}dx \\
&\le 2\frac{m^3}{\log m} \sum_{l=1}^{[\sqrt{m}]} \frac{1}{l^3} \sin\frac{(2l+1)\pi}{m} \left(\cos\frac{2l\pi}{m} - \cos\frac{(2l+1)\pi}{m}\right) \\
&= 4\,\frac{m^3}{\log m} \sum_{l=1}^{[\sqrt{m}]} \frac{1}{l^3} \sin\frac{(2l+1)\pi}{m} \sin\frac{\pi}{2m} \sin\frac{(4l+1)\pi}{2m} \\
&\le \frac{C_5}{\log m} \sum_{l=1}^{[\sqrt{m}]} \frac{1}{l} \le C_6 \quad \text{for} \quad m \ge m_0^{(1)},
\end{aligned}$$

where $m_0^{(1)}$, C_5, C_6 are absolute positive constants. Now we prove that for $m \ge m_0^{(2)}$ the following inequality is true

$$\max_{0\le k\le m} |a_{m-1}(Q_{m-1},k,\varphi)| \ge C_3 (\log m)^{\frac{1}{3}} \quad \text{for all real} \quad \varphi, \tag{23}$$

where C_3 and $m_0^{(2)}$ are absolute positive constants. Indeed, it is known [2] that if $F \in L^2_w([-1,1])$, $\quad w(t) = 2\sqrt{1-t^2}, \quad t \in [-1,1],$ then for the function

$$P(x,y) := F(x) \quad (x,y) \in \mathbb{B}^2 \tag{24}$$

we have

$$a_m(P,\, k,\, \varphi) := \frac{\sqrt{\pi}}{m+1} \hat{F}(m) u_m \left(\cos \left(\frac{k\pi}{m+1} + \varphi \right) \right) \tag{25}$$

where $k = 0,\, 1,\, \ldots\, m,\ \varphi \in (-\infty, \infty)$ and

$$\hat{F}(m) := 2 \int_{-1}^{1} F(t) u_m(t) \sqrt{1-t^2} dt\,. \tag{26}$$

Further we show that for some absolute positive constant C_7

$$|\hat{q}_{(m-1)}(m-1)| \geq\, C_7 (\log m)^{\frac{1}{3}}. \tag{27}$$

According to (19), (20), (26) we get

$$\begin{aligned}
&\hat{q}_{(m-1)}(m-1) := 2 \int_{-1}^{1} q_{m-1}(t) u_{m-1}(t) \sqrt{1-t^2} dt \\
&= 2 \, \frac{m^2}{(\log m)^{\frac{2}{3}}} \sum_{k=1}^{[\sqrt{m}]} \int_{-1}^{1} f_k^{(m)}(t) u_{m-1}(t) \sqrt{1-t^2} dt \\
&= 2 \, \frac{m^2}{(\log m)^{\frac{2}{3}}} \sum_{k=1}^{[\sqrt{m}]} \frac{1}{k^2} \int_{\cos \frac{(2k+1)\pi}{m}}^{\cos \frac{2k\pi}{m}} \sin m \arccos t dt \\
&\geq 2 \, \frac{m^2}{(\log m)^{\frac{2}{3}}} \sum_{k=1}^{[\sqrt{m}]} \frac{1}{k^2} \int_{\frac{2k\pi}{m}}^{\frac{(2k+1)\pi}{m}} \sin m\vartheta \sin \vartheta d\vartheta \\
&\geq\, C_8 \, \frac{1}{(\log m)^{\frac{2}{3}}} \sum_{k=1}^{[\sqrt{m}]} \frac{1}{k} \,\geq\, C_7 (\log m)^{\frac{1}{3}}\,,
\end{aligned}$$

where C_7 and C_8 are positive absolute constants. Let φ_0 be such that $\varphi_0 = \varphi \pmod{\pi}$ and $0 \leq \varphi_0 < \pi$. Now we prove that there exists an integer $k_1 = k_1(\varphi_0)$ with the properties

$$\left| u_{m-1} \left(\cos \left(\frac{k_1 \pi}{m} + \varphi_0 \right) \right) \right| \geq \frac{2}{\pi} m \quad \text{and} \quad 0 \leq k_1 \leq m-1. \tag{28}$$

Consider the following cases: let first $0 \le \varphi_0 < \frac{\pi}{2m}$. In this case we take $k_1 := 0$. We see that then (cf.(2))

$$\left| u_{m-1}\left(\cos\left(\frac{k_1\pi}{m} + \varphi_0\right)\right)\right| = |u_{m-1}(\cos\varphi_0)| \, \frac{|\sin m \arccos(\cos\varphi_0)|}{|\sin\varphi_0|} \ge \frac{2}{\pi} m \, .$$

If now $\frac{\pi}{2m} \le \varphi_0 < \frac{\pi}{m}$ then we choose $k_1 = m-1$. It is clear that then

$$\begin{aligned} &\left| u_{m-1}\left(\cos\left(\frac{k_1\pi}{m} + \varphi_0\right)\right)\right| = \left| u_{m-1}\left(\cos\left(\frac{(m-1)\pi}{m} + \varphi_0\right)\right)\right| \\ &= \left| u_{m-1}\left(-\cos\left(\frac{\pi}{m} - \varphi_0\right)\right)\right| = \left| u_{m-1}\left(\cos\left(\frac{\pi}{m} - \varphi_0\right)\right)\right| \\ &= \frac{|\sin m\left(\frac{\pi}{m} - \varphi_0\right)|}{|\sin\left(\frac{\pi}{m} - \varphi_0\right)|} \ge \frac{2}{\pi} m \, . \end{aligned}$$

Now it remains only the case $\frac{\pi}{m} \le \varphi_0 < \pi$. Let $k_0 = k_0(\varphi_0)$ be the integer such that

$$\frac{k_0\pi}{m} + \varphi_0 < \pi \le \frac{(k_0+1)\pi}{m} + \varphi_0 \, . \tag{29}$$

It is clear that in this case (cf.(29), (30))

$$\frac{k_0\pi}{m} < \pi - \varphi_0 \le \pi - \frac{\pi}{m} \quad \text{and} \quad \frac{(k_0+1)\pi}{m} \ge \pi - \varphi_0 > 0 \, ,$$

and consequently,

$$0 \le k_0 < m-1 \quad \text{and} \quad 0 < \pi - \left(\frac{k_0\pi}{m} + \varphi_0\right) \le \frac{\pi}{m}. \tag{30}$$

Now we have two subcases:

$$0 < \pi - \left(\frac{k_0\pi}{m} + \varphi_0\right) \le \frac{\pi}{2m} \tag{31}$$

and

$$\frac{\pi}{2m} < \pi - \left(\frac{k_0\pi}{m} + \varphi_0\right) \le \frac{\pi}{m}. \tag{32}$$

Let

$$k_1 := k_0 \quad \text{in the first subcase and} \quad k_1 := k_0 + 1 \quad \text{in the second subcase.} \tag{33}$$

It is clear that in both cases(cf.(30))

$$0 \le k_1 \le m-1.$$

In the first subcase we have (cf.(2), (33), (31))

$$\left| u_{m-1}\left(\cos\left(\frac{k_1\pi}{m}+\varphi_0\right)\right)\right| = \left| u_{m-1}\left(\cos\left(\frac{k_0\pi}{m}+\varphi_0\right)\right)\right|$$

$$= \left| u_{m-1}\left(\cos\left(\pi-\left(\frac{k_0\pi}{m}+\varphi_0\right)\right)\right)\right| = \frac{\left|\sin m\left(\pi-\left(\frac{k_0\pi}{m}+\varphi_0\right)\right)\right|}{\left|\sin\left(\pi-\left(\frac{k_0\pi}{m}+\varphi_0\right)\right)\right|} \ge \frac{2}{\pi}m.$$

And at last for the second subcase we get (cf.(13), (33), (32))

$$\left| u_{m-1}\left(\cos\left(\frac{k_1\pi}{m}+\varphi_0\right)\right)\right| = \left| u_{m-1}\left(\cos\left(\frac{(k_0+1)\pi}{m}+\varphi_0\right)\right)\right|$$

$$= \left| u_{m-1}\left(\cos\left(\frac{(k_0+1)\pi}{m}+\varphi_0-\pi\right)\right)\right| = \frac{\left|\sin m\left(\left(\frac{(k_0+1)\pi}{m}+\varphi_0-\pi\right)\right)\right|}{\left|\sin\left(\left(\frac{(k_0+1)\pi}{m}+\varphi_0-\pi\right)\right)\right|} \ge \frac{2}{\pi}m.$$

The inequality (28) and consequently the inequality (23) are proved. now we will estimate $\omega_1(\delta; Q_{m-1})_{\frac{3}{2}}$. Taking account of the fact that for $k = 1,\, 2,\, \ldots\, [\sqrt{m}]$,

$$\cos\frac{(2k+1)\pi}{m} - \cos\frac{(2k+2)\pi}{m} \ge 2\sin\frac{\pi}{2m}\sin\frac{(4k+3)\pi}{2m} \ge \frac{2}{m^2},$$

we get for $|h| \le \frac{2}{m^2}$ (cf.(19), (20), (21))

$$\int_{\mathbb{B}^2\bigcap\mathbb{B}^2(1,h)} |Q_{m-1}(x+h,y)-Q_{m-1}(x,y)|^{\frac{3}{2}}\,dxdy \;\le\; |h|\,\frac{m^3}{\log m}\sum_{k=1}^{[\sqrt{m}]}\frac{1}{k^3}\sin\frac{(2k+2)\pi}{m}$$

$$\le\; C_9\,|h|\,\frac{m^2}{\log m}\sum_{k=1}^{[\sqrt{m}]}\frac{1}{k^2} \;\le\; C_{10}|h|\,\frac{m^2}{\log m}\,,$$

and for $|h| > \frac{2}{m^2}$ we have (cf.(22))

$$\int_{\mathbb{B}^2\bigcap\mathbb{B}^2(1,h)} |Q_{m-1}(x+h,y)-Q_{m-1}(x,y)|^{\frac{3}{2}}\,dxdy \;\le\; \left(2\|Q_{m-1}\|_{\frac{3}{2}}\right)^{\frac{3}{2}} \;\le\; C_{11},$$

for some absolute C_9, C_{10} and C_{11}. From (19), (20), (21) we see that the Lemma is established completely.

Proof of Theorem 2 . We define an increasing sequence of positive integers $\{m_l\}_{l=1}^{\infty}$ by induction. Let $m_1 = m_0 + 1$ where m_0 is the number from the Lemma. Now let numbers $m_1, m_2 \ldots m_{l-1}$ be already defined. Introduce the functions defined on $\mathbb{B}^2$ and [-1,1] correspondingly

$$A_{l-1}(x,y) := \sum_{k=1}^{l-1} \frac{1}{(\log m_k)^{\frac{1}{3}}} Q_{m_k-1}(x,y)\,, \tag{34}$$

and

$$B_{l-1}(x) := \sum_{k=1}^{l-1} \frac{1}{(\log m_k)^{\frac{1}{3}}} q_{m_k-1}(x)\,, \tag{35}$$

where $Q_{m_k-1}(x,y)$ and $q_{m_k-1}(x)$ are functions from the Lemma corresponding to the number m_k. It is clear that (cf.(24), (19), (20)) $A_{l-1}(x,y) \in L^2(\mathbb{B}^2)\,,\ B_{l-1}(x) \in L^2([-1,1])$ and (cf. (24), (25), (26))

$$|a_{m-1}(A_{l-1},\, k,\, \varphi)| \leq \pi |\hat{B}_{l-1}(m-1)|\,, \text{for all real}\, \varphi\,.$$

It is clear that

$$\lim_{m\to\infty} |\hat{B}_{l-1}(m-1)| = 0.$$

¿From the last equation we conclude that there is the number N_{l-1} such that for all $m \geq N_{l-1}$

$$|\hat{B}_{l-1}(m-1)| \leq \frac{C_3}{2\pi} \tag{36}$$

where C_3 is the constant from the Lemma. Now we define m_l so that the following relations are satisfied:

$$m_l > m_{l-1}, \quad m_l \geq N_{l-1}, \tag{37}$$

$$\frac{m_{l-1}}{(\log m_l)^{\frac{1}{3}}} \leq \frac{1}{l+1}\,, \tag{38}$$

$$2(\log m_l)^{-\frac{1}{3}} \leq (\log m_{l-1})^{-\frac{1}{3}}, \tag{39}$$

and

$$\frac{m_l^{\frac{4}{3}}}{\log m_l} \geq 2\frac{m_{l-1}^{\frac{4}{3}}}{\log m_{l-1}}. \tag{40}$$

Thus we have constructed the infinite increasing sequence of integers $\{m_l\}_{l=1}^{\infty}$. Consider the function

$$g(x,y) := \sum_{k=1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} \, Q_{m_k-1}(x,y) \tag{41}$$

defined on $\mathbb{B}^2$. It is obvious that (cf.(41), (22), (39))

$$\|g\|_{\frac{3}{2}} \le \sum_{k=1}^{\infty} \frac{C_4}{(\log m_k)^{\frac{1}{3}}} < \infty \, . \tag{42}$$

Let $\{\varphi_m\}_{m=0}^{\infty}$ be an arbitrary sequence of real numbers. According to (34), (41) we get for each $k = 0,\, 1,\, \dots\; m_l - 1,\; l = 1,\, 2,\, \dots\,,$

$$\begin{aligned} & a_{m_l-1}(g,\, k,\, \varphi_{m_l-1}) = a_{m_l-1}(A_{l-1},\, k,\, \varphi_{m_l-1}) + a_{m_l-1}(Q_{m_l-1}(\log m_l)^{-\frac{1}{3}},\, k,\, \varphi_{m_l-1}) \\ + \; & a_{m_l-1}(E_l,\, k,\, \varphi_{m_l-1}), \end{aligned}$$

where

$$E_l(x,y) := \sum_{k=l+1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} \, Q_{m_k-1}(x,y) \, . \tag{43}$$

According to (25), (34), (35), (36), (26), (37) for each $k = 0,\, 1,\, \dots\; m_l - 1,\; l = 1,\, 2,\, \dots$ the following inequality holds true

$$|a_{m_l-1}(A_{l-1},\, k,\, \varphi_{m_l-1})| \le \frac{C_3}{2} \, . \tag{44}$$

On the other hand, it follows from (39), (15), (38) and (43) that for each $k = 0,\, 1,\, \dots\,,\, m_l - 1,$ $l = 1,\, 2,\, \dots\,,$ we have

$$|a_{m_l-1}(E_l,\, k,\, \varphi_{m_l-1})| = O\left(\sum_{k=l+1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}}\right) = O\left(\frac{1}{l+1}\right) \quad \text{as} \quad l \to \infty. \tag{45}$$

Now it is easy to see that (cf. (45), (44)) for each $k = 0,\, 1,\, \dots\; m_l - 1,\; l = 1,\, 2,\, \dots,$ we get

$$\begin{aligned} & |a_{m_l-1}(Q_{m_l-1}(\log m_l)^{-\frac{1}{3}},\, k,\, \varphi_{m_l-1})| \le |a_{m_l-1}(g,\, k,\, \varphi_{m_l-1})| \\ & + |a_{m_l-1}(A_{l-1},\, k,\, \varphi_{m_l-1})| + |a_{m_l-1}(E_l,\, k,\, \varphi_{m_l-1})| \le |a_{m_l-1}(g,\, k,\, \varphi_{m_l-1})| \\ & + \frac{C_3}{2} + O\left(\frac{1}{l+1}\right) \le \max_{0 \le k \le m_l-1} |a_{m_l-1}(g, k, \varphi_{m_l-1})| \\ & + \frac{C_3}{2} + O\left(\frac{1}{l+1}\right) \quad \text{as } l \to \infty \end{aligned}$$

and therefore according to the Lemma (cf.(14))

$$\max_{0\le k\le m_l-1} |a_{m_l-1}(g,k,\varphi_{m_l-1})| \ \ge \max_{0\le k\le m_l-1} |a_{m_l-1}(Q_{m_l-1}(\log m_l)^{-\frac{1}{3}},\ k,\ \varphi_{m_l-1})|$$
$$\frac{C_3}{2} - O\left(\frac{1}{l+1}\right) \ \ge\ C_3 - \frac{C_3}{2} - o(1) \quad \text{as} \quad l\to\infty.$$

We see now that the relation (10) of theorem 2 is established. it is obvious from (13), (41) and the Lemma that the function $g(x,y)$ is in fact a function of one variable and consequently the second equation in (17) is true. It remains only to estimate $\omega_1(\delta;g)_{\frac{3}{2}}$. Let for a given $\delta>0$ the integer $l_0 = l_o(\delta)$ be such that

$$\frac{2}{m^2_{l_0+1}} < \delta \le \frac{2}{m^2_{l_0}}\,.$$

¿From (16), (41), (40), (39) we see that

$$\omega_1(\delta;g)_{\frac{3}{2}} \ \le \sum_{k=1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}}\,\omega_{m_k-1}(\delta) \ \le\ C_5\,\delta^{\frac{2}{3}} \sum_{k=1}^{l_0} \frac{m_k^{\frac{4}{3}}}{\log m_k}$$
$$+\,2C_5 \sum_{k=l_0+1}^{\infty} \frac{1}{(\log m_k)^{\frac{1}{3}}} \ \le\ 2C_5 \frac{m_{l_0}^{\frac{4}{3}}}{\log m_{l_0}}\delta^{\frac{2}{3}} + 4C_5\,\frac{1}{(\log m_{l_0+1})^{\frac{1}{3}}} \le C_{12}\,\frac{1}{\log\frac{1}{\delta}}$$
$$+C_{13}\,\frac{1}{\left(\log\frac{1}{\delta}\right)^{\frac{1}{3}}} = O\left(\frac{1}{\left(\log\frac{1}{\delta}\right)^{\frac{1}{3}}}\right) \quad \text{as}\delta \quad \to 0+\,.$$

Theorem 2 is now proven.

References

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99

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