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ON TWO PROBLEMS IN THE MULTIVARIATE APPROXIMATION[1]

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Abstract. The paper contains two theorems on approximation of functions with bounbed mixed derivative. These theorems give some progress in two old open problems. The first one gives, in particular, an upper estimate in the Bernstein L_1-inequality for trigonometric polynomials on two variables with harmonics in hyperbolic crosses. The second one gives the order of the entropy numbers and Kolmogorov's widths in the L_∞-norm of the class $MW^r_{\infty,\alpha}$ of functions of two variables.

1. Introduction

In this paper we report some progress in two old problems from approximation of multivariate functions with bounded mixed derivative. These two problems originate from the very first publications (see [B1], [B2], [M], [Te]) on the subject. We introduce first some notations. We give these notations in the general d-dimensional (functions on d variables) case and point out that the results in this paper concern only the 2-dimensional case. Let $s = (s_1, \dots, s_d)$ be a vector whose coordinates are nonnegative integers and

$$\rho(s) := \{k \in Z^d : [2^{s_j-1}] \le |k_j| < 2^{s_j}, j = 1, \dots, d\},$$

where $[a]$ means the integer part of a number a. Denote

$$Q_m := \bigcup_{\|s\|_1 \le m} \rho(s)$$

and denote by $\mathcal{T}(Q_m)$ the subspace of trigonometric polynomials with harmonics in Q_m.

Problem 1. (The Bernstein inequality in L_1). *Find the order in m of the sequence*

$$b(m, r, L_1) := \sup_{f \in \mathcal{T}(Q_m)} \|f^{(r,\dots,r)}\|_1 / \|f\|_1, \quad m = 1, 2 \dots .$$

This problem is open for all $r > 0$ and $d \ge 2$. The corresponding problem for the L_p-norm, $1 < p \le \infty$, was solved in early 60-th. K.I. Babenko [B2] proved the upper bound

$$b(m, r, L_\infty) \ll m^{d-1} 2^{rm}$$

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and S.A. Telyakovskii [Te] proved the lower bound

$$b(m, r, L_\infty) \gg m^{d-1} 2^{rm}.$$

B.S. Mityagin [M] proved for $1 < p < \infty$

$$b(m, r, L_p) \asymp 2^{rm}.$$

For these results see also [T1], Ch.1, s.1. The known bounds for $b(m, r, L_1)$ are simple

$$2^{rm} \ll b(m, r, L_1) \ll m^{d-1} 2^{rm}.$$

We prove here a new upper bound in the 2-dimensional case:

$$b(m, r, L_1) \ll m^{1/2} 2^{rm}.$$

The second problem we are dealing with here is about the Kolmogorov widths of classes $MW^r_{\infty,\alpha}$ in the L_∞-norm. Let for $r > 0$ and $\alpha \in \mathbb{R}$

$$F_r(t, \alpha) = 1 + 2 \sum_{k=1}^{\infty} k^{-r} \cos(kt - \alpha\pi/2), \quad t \in [0, 2\pi],$$

and for $x = (x_1, \dots, x_d)$ and $\alpha = (\alpha_1, \dots, \alpha_d)$

$$F_r(x, \alpha) = \prod_{j=1}^{d} F_r(x_j, \alpha_j).$$

Define

$$MW^r_{q,\alpha} = \{f : f = F_r(\cdot, \alpha) * \phi(\cdot), \quad \|\phi\|_q \le 1\},$$

where $*$ means the convolution. We recall the definition of the Kolmogorov width of a centrally symmetric set A in a Banach space X

$$d_n(A, X) := \inf_{g_j \in X, j=1,\dots,n} \sup_{f \in A} \inf_{c_j, j=1,\dots,n} \|f - \sum_{j=1}^{n} c_j g_j\|_X.$$

Problem 2. *Find the order of the sequence* $\{d_n(MW^r_{\infty,\alpha}, L_\infty)\}_{n=1}^{\infty}$.

The first upper bounds in this problem were obtained by K.I. Babenko [B2]

$$d_n(MW^r_{\infty,\alpha}, L_\infty) \ll n^{-r} (\log n)^{(d-1)(r+1)}.$$

We prove in this paper the lower bound

$$d_n(MW^r_{\infty,\alpha}, L_\infty) \gg n^{-r} (\log n)^{r+1/2}, \quad r > 1/2,$$

in the 2-dimensional case. This lower estimate combined with the corresponding known upper estimate (see [Be]) gives the solution to Problem 2 in the case $d = 2$ and $r > 1/2$:

$$d_n(MW^r_{\infty,\alpha}, L_\infty) \asymp n^{-r} (\log n)^{r+1/2}.$$

Problem 2 is open in the case $d > 2$.

2. The Bernstein inequalities

We introduce some more notations. Let for two integers $a \ge 1$ and $0 \le b < a$ $AP(a,b)$ denote an arithmetic progression of the form $al+b$, $l = 0,1,\dots$ and

$$H_n(a,b) := \{s : s \in \mathbb{Z}^2_+, \quad \|s\|_1 = n, \quad s_1, s_2 \ge a, \quad s_1 \in AP(a,b)\}.$$

It will be convenient for us to consider subspaces $\mathcal{T}(\rho'(s))$ of trigonometric polynomials with harmonics in

$$\rho'(s) := \{k \in \mathbb{Z}^2 : [2^{s_j-2}] \le |k_j| < 2^{s_j}, \quad j = 1,2\}.$$

For a subspace Y in $L_2(\mathbb{T}^d)$ we denote by $Y^\perp$ its orthogonal complement.

Lemma 2.1. *Take any trigonometric polynomials $t_s \in \mathcal{T}(\rho'(s))$ and form the function*

$$\Phi(x) := \prod_{s \in H_n(a,b)} (1 + t_s(x)).$$

Then for any $a \ge 6$ and any $0 \le b < a$ this function admits the representation

$$\Phi(x) = 1 + \sum_{s \in H_n(a,b)} t_s(x) + R(x)$$

with $R \in \mathcal{T}(Q_{n+a-6})^\perp$.

The proof of this lemma is similar to the proof of Lemma 2.1 from [T2] (for the idea of the proof see also [T1], p. 57(60)).

Remark 2.1. *For any real numbers $|y_l| \le 1$, $l = 1,\dots,N$, we have ($i^2 = -1$)*

$$|\prod_{l=1}^{N}(1 + \frac{iy_l}{\sqrt{N}})| \le C.$$

Lemma 2.2. *For any function f of the form*

$$f = \sum_{s \in H_n(a,b)} t_s$$

with $a \ge 6$, $0 \le b < a$, and t_s, $s \in H_n(a,b)$, is a real trigonometric polynomial in $\mathcal{T}(\rho'(s))$ such that $\|t_s\|_\infty \le 1$ we have

$$E^\perp_{Q_{n+a-6}}(f)_\infty := \inf_{g \in \mathcal{T}(Q_{n+a-6})^\perp} \|f - g\|_\infty \ll (1 + n/a)^{1/2}.$$

Proof. Let us form the function

$$RP(f) := \operatorname{Im} \prod_{s \in H_n(a,b)} (1 + it_s(1+n/a)^{-1/2}),$$

which is an analog of the Riesz product. Then by Remark 2.1 we have

$$\|RP(f)\|_\infty \le C. \tag{2.1}$$

Lemma 2.1 provides the representation

$$RP(f) = (1+n/a)^{-1/2} \sum_{s \in H_n(a,b)} t_s + g, \quad g \in \mathcal{T}(Q_{n+a-6})^\perp. \tag{2.2}$$

Combining (2.1) and (2.2) we get the statement of Lemma 2.2.

Remark 2.2. *It is clear that in Lemma 2.2 we can drop the assumption that the t_s are real polynomials.*

Lemma 2.3. *For any function f of the form*

$$f = \sum_{\|s\|_1 = n} t_s, \quad t_s \in \mathcal{T}(\rho'(s)), \quad \|t_s\|_\infty \le 1,$$

we have for any $a \ge 6$

$$E^{\perp}_{Q_{n+a-6}}(f)_\infty \le Ca(1+n/a)^{1/2}.$$

Proof. Let us introduce some more notations. Denote

$$\theta_n := \{s : \|s\|_1 = n\}; \qquad \theta_{n,a} := \{s \in \theta_n, \quad s_1 < a \quad \text{or} \quad s_2 < a\}.$$

Then

$$f = \sum_{s \in \theta_n} t_s = \sum_{s \in \theta_{n,a}} t_s + \sum_{b=0}^{a-1} \sum_{s \in H_n(a,b)} t_s$$

and

$$E^{\perp}_{Q_{n+a-6}}(f)_\infty \le \sum_{s \in \theta_{n,a}} \|t_s\|_\infty + \sum_{b=0}^{a-1} E^{\perp}_{Q_{n+a-6}}(\sum_{s \in H_n(a,b)} t_s)_\infty.$$

Using the assumption $\|t_s\|_\infty \le 1$, Lemma 2.2 and Remark 2.2 we get from here the required estimate.

Lemma 2.3 is proved.

It proved to be useful in studying approximation of functions with bounded mixed derivative to consider along with the L_p-norms the Besov type norms. Let $V_n(t)$ be the de la Vallée-Poussin polynomials, $t \in [0, 2\pi]$. We define

$$\mathcal{A}_0(t) := 1, \quad \mathcal{A}_1(t) := V_1(t) - 1, \quad \mathcal{A}_n(t) := V_{2^{n-1}}(t) - V_{2^{n-2}}(t), \quad n \ge 2,$$

and for $x = (x_1, x_2)$, $s = (s_1, s_2)$

$$\mathcal{A}_s(x) := \mathcal{A}_{s_1}(x_1)\mathcal{A}_{s_2}(x_2).$$

Consider the convolution operator A_s with the kernel $\mathcal{A}_s(x)$,

$$A_s(f) := f * \mathcal{A}_s,$$

and define the $B^r_{1,1}$-norm as follows

$$\|f\|_{B^r_{1,1}} := \sum_s 2^{r\|s\|_1} \|A_s(f)\|_1.$$

Theorem 2.1. *Let $r > 0$ be given. For any $f \in \mathcal{T}(Q_m)$ we have*

$$\|f\|_{B^r_{1,1}} \le C(r) m^{1/2} 2^{rm} \|f\|_1.$$

Proof. Take any $n \le m+2$ and consider the sum

$$S_n := \sum_{\|s\|_1 = n} \|A_s(f)\|_1.$$

Define the polynomials $t_s := A_s(\operatorname{sign} A_s(f))$ and

$$g_n := \sum_{\|s\|_1 = n} \bar{t_s}.$$

It is clear from the definition of $A_s(\cdot)$ that $\bar{t_s} \in \mathcal{T}(\rho'(s))$. Next, on one hand we have

$$\langle f, g_n \rangle = \sum_{\|s\|_1 = n} \|A_s(f)\|_1 \tag{2.3}$$

and on the other hand we have

$$\langle f, g_n \rangle \le \|f\|_1 E^{\perp}_{Q_m}(g_n)_\infty. \tag{2.4}$$

By Lemma 2.3 with $a = \max(m - n + 6, 6)$ we get

$$E^{\perp}_{Q_m}(g_n)_\infty \ll (m - n + 6)(1 + n/(m - n + 6))^{1/2}. \tag{2.5}$$

The relations (2.3)–(2.5) imply the inequality

$$\sum_s 2^{r\|s\|_1} \|A_s(f)\|_1 = \sum_{n \le m+2} 2^{rn} S_n \le C(r) m^{1/2} 2^{rm} \|f\|_1,$$

that is what was required.

We consider a more general derivative than standard mixed derivative. For a polynomial $f \in \mathcal{T}(Q_m)$ we define its (r,α)-derivative as the convolution with the kernel

$$U^r_m(x,\alpha) := 4 \sum_{k \in Q_m, k > 0} (k_1 k_2)^r \cos(k_1 x_1 + \alpha_1 \pi/2) \cos(k_2 x_2 + \alpha_2 \pi/2).$$

Thus,

$$f^{(r)}(x,\alpha) := D^{r,\alpha} f := f(\cdot) * U^r_m(\cdot,\alpha).$$

Corollary 2.1. *For any $r > 0$, $\alpha \in \mathbb{R}^2$, we have for any function $f \in \mathcal{T}(Q_m)$ on two variables the inequality*

$$\|f^{(r)}(x,\alpha)\|_1 \le C(r)m^{1/2}2^{rm}\|f\|_1.$$

Proof. By Bernstein inequality for trigonometric polynomials with harmonics in rectangles we get

$$\|f^{(r)}(x,\alpha)\|_1 \le \sum_s \|D^{r,\alpha}A_s(f)\|_1 \le C(r)\sum_s 2^{r\|s\|_1}\|A_s(f)\|_1 \le$$

and using Theorem 2.1 we continue

$$\le C(r)m^{1/2}2^{rm}\|f\|_1.$$

Remark 2.3. *The inequality in Theorem 2.1 is sharp. The multiplier $m^{1/2}2^{rm}$ can not be replaced by smaller function on m even if we write $\|f\|_p$, $p < \infty$, instead of $\|f\|_1$ in Theorem 2.1.*

Proof. Indeed, take

$$f = \sum_{\|s\|_1 = m} \cos 2^{s_1}x_1 \cos 2^{s_2}x_2.$$

Then

$$\sum_s 2^{r\|s\|_1}\|A_s(f)\|_1 \asymp m2^{rm}$$

and by Littlewood-Paley Theorem

$$\|f\|_p \ll m^{1/2}, \quad p < \infty.$$

The next theorem shows that we can improve the inequality in Theorem 2.1 if replace $\|f\|_1$ by $\|f\|_\infty$.

Theorem 2.2. *Let $r > 0$ be given. For any $f \in \mathcal{T}(Q_m)$ we have*

$$\|f\|_{B^r_{1,1}} \le C(r)2^{rm}\|f\|_\infty.$$

Proof. Using functions $\Phi(x)$ defined in Lemma 2.1 and their property $\|\Phi\|_1 = 1$ instead of functions $RP(f)$ defined in the proof of Lemma 2.2 and their property $\|RP(f)\|_\infty \le C$ we get the following analog of Lemma 2.3 (see also Lemma 2.1 in [T2]).

Lemma 2.4. *For any function f of the form*

$$f = \sum_{\|s\|_1 = n} t_s, \quad t_s \in \mathcal{T}(\rho'(s)), \quad \|t_s\|_\infty \le 1,$$

we have for any $a \ge 6$

$$E^\perp_{Q_{n+a-6}}(f)_1 \le Ca.$$

We repeat now the arguments from the proof of Theorem 2.1 with (2.4) replaced by

$$\langle f, g_n\rangle \le \|f\|_\infty E^\perp_{Q_m}(g_n)_1$$

and Lemma 2.3 replaced by Lemma 2.4.

3. The Kolmogorov widths

We prove in this section estimates for the entropy numbers and Kolmogorov's widths of the class $MW^r_{\infty,\alpha}$. We use the following definition of the entropy numbers

$$\epsilon_m(F,X) = \inf\{\epsilon : \exists f_1,\dots,f_{2^m} \in X : F \subset \cup_{j=1}^{2^m}(f_j + \epsilon B(X))\},$$

where $B(X)$ is the unit ball of the Banach space X .

Theorem 3.1. *Let $r > 1/2$. We have for the class $MW^r_{\infty,\alpha}$ of functions on two variables the asymptotic relations*

$$\epsilon_m(MW^r_{\infty,\alpha}, L_\infty) \asymp m^{-r}(\log m)^{r+1/2},$$

$$d_m(MW^r_{\infty,\alpha}, L_\infty) \asymp m^{-r}(\log m)^{r+1/2}.$$

Proof. The upper estimates are known (see [Be]). We prove here the corresponding lower estimates. We consider first the entropy numbers and use the interpolation property of these numbers (see [P], s. 12.1.12)

$$\epsilon_{2m}(MW^r_{4,\alpha}, L_\infty) \le 2\epsilon_m(MW^r_{2,\alpha}, L_\infty)^{1/2}\epsilon_m(MW^r_{\infty,\alpha}, L_\infty)^{1/2}. \tag{3.1}$$

Next, from [T2] we get for $r > 1/4$

$$\epsilon_{2m}(MW^r_{4,\alpha}, L_\infty) \ge C(r)m^{-r}(\log m)^{r+1/2}$$

and from [Be] we have for $r > 1/2$

$$\epsilon_m(MW^r_{2,\alpha}, L_\infty) \le C(r)m^{-r}(\log m)^{r+1/2}.$$

Substituting these two estimates into (3.1) we get the required lower estimate for the entropy numbers.

It is known (see [L]) that the entropy numbers give in a certain sense the lower bounds for the Kolmogorov widths. We formulate this in a way convenient for us (see [KT] and also [T3]).

Lemma 3.1. *Let A be a compact set in a separable Banach space X. Assume that for two real numbers $r > 0$ and b we have*

$$\epsilon_m(A,X) \asymp m^{-r}(\log m)^b.$$

Then for Kolmogorov's widths of this set we have

$$d_m(A,X) \gg m^{-r}(\log m)^b.$$

Application of this lemma completes the proof of Theorem 3.1.

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