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K.I. Oskolkov

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Department of Mathematics
University of South Carolina

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Our goal is to compare the efficiencies of linear and nonlinear methods in the problem of ridge approximation. We confine ourselves by functions of two variables $f(\mathbf{x}) = f(x_1, x_2)$ and the norm $\|\cdot\|$ of Hilbert space $\mathcal{L}^2(\mathbb{B}^2)$, where $\mathbb{B}^2$ is the unit disc $|\mathbf{x}| \leq 1$ on the plane $\mathbb{R}^2$. By definition,

$$\mathcal{R}_N^{\text{free}}(f) := \inf_{R \in \mathcal{W}_N^{\text{free}}} \|f - R\|, \quad \mathcal{R}_N^{\text{equi}}(f) := \min_{R \in \mathcal{W}_N^{\text{equi}}} \|f - R\|;$$

$\mathcal{W}_N^{\text{free}}$ – the set of all $N$-terms linear combinations of functions of the planar wave type, $R(\mathbf{x}) = \sum_1^N F_j(\mathbf{x} \cdot \boldsymbol{\theta}_j)$; $\mathbf{x} \cdot \mathbf{y}$ denotes the usual inner product. Wave profiles $F_j(x)$, $x \in \mathbb{R}^1$ are subject to optimization in both cases and depend upon the given $f$. The sets $\{\boldsymbol{\theta}_j = \langle \cos\vartheta_j, \sin\vartheta_j \rangle\}_1^N$ (wave vectors) are beeing optimized only in the problem $\mathcal{R}^{\text{free}}$; the angles $\vartheta_j$ are equispaced in $\mathcal{R}^{\text{equi}}$, and $\vartheta_j := \frac{\pi j}{N}$.

The problem $\mathcal{R}^{\text{free}}$ is of a stronly nonlinear nature, due to optimization in $\{\boldsymbol{\theta}_j\}_1^N \subset \mathcal{S}^1$. In particular, simple examples demonstrate (cf. [1]), that for $N \geq 2$ the infimum may be not reachable (ill-posedness). On the contrary, the problem of $\mathcal{R}^{\text{equi}}$ is linear and solved by orthogonal projection in $\mathcal{L}^2(\mathbb{B}^2)$ onto the corresponding subspace.

Further, let $\mathcal{P}_N^1 := \operatorname{Span}\{x^k\}_{k \leq N}$, $\mathcal{P}_N^2 := \operatorname{Span}\{x_1^k x_2^l\}_{k+l \leq N}$. It is known (cf. [2], [3]), that if $\{\boldsymbol{\theta}_j\}_1^N$ is an arbitrary set of $N$ directions, where the corresponding angles $\vartheta_j$ are pairwise non-congruent $\bmod \pi$, then each polynomial $P(\mathbf{x}) \in \mathcal{P}_{N-1}^2$ is a linear combination of $N$ planar wave polynomials, $P(\mathbf{x}) = \sum_1^N P_j(\mathbf{x} \cdot \boldsymbol{\theta}_j)$, where $P_j \in \mathcal{P}_{N-1}^1$. Thus, the best algebraic approximations $\mathcal{E}_N^{\text{poly}}(f) := \min_{P \in \mathcal{P}_{N-1}^2} \|f - P\|$ provide a natural intermediate characterization for ridge approximation. In particular, $\mathcal{R}_N^{\text{free}}(f) \leq \mathcal{R}_N^{\text{equi}}(f) \leq \mathcal{E}_N^{\text{poly}}(f)$.

One of the interesting problems: when the nonlinear method $\mathcal{R}^{\text{free}}$ is more effective in the order than the linear one $\mathcal{R}^{\text{equi}}$, i .e. , when $\mathcal{R}_N^{\text{free}}(f) = o\left(\mathcal{R}_N^{\text{equi}}(f)\right)$, $N \to \infty$ ?

The following statement contains the solution of this problem in two model cases: a) radial $f(\mathbf{x}) = f(|\mathbf{x}|)$, and b) harmonic functions in the open disc, $\Delta f(\mathbf{x}) = 0$, $|\mathbf{x}| < 1$ (notations: $f =$

$f_{\rm harm}$, $f = f_{\rm rad}$). For the sake of brievity, we state only order type results, although in several cases the expicit numerical values of the constants are also known. We use the notation $A << B$, iff there is a universal positive constant $c$ such that $\frac{1}{c}B \le A \le cB$; if $B << A << B$, we will write $A \sim B$.

**Theorem 1** *The following relations hold true*

$$f = f_{\rm rad} \Longrightarrow \mathcal{R}_N^{\rm equi}(f) \sim \mathcal{E}_{2N}^{\rm poly}(f); \qquad f = f_{\rm harm} \Longrightarrow \mathcal{E}_{N+1}^{\rm poly}(f) << \mathcal{R}_N^{\rm equi}(f) \le \mathcal{E}_N^{\rm poly}(f); \tag{1}$$

$$f = f_{\rm rad} \Longrightarrow \mathcal{R}_N^{\rm free}(f) \ge \sup_{M\ge N} \sqrt{\frac{M-N}{2M}}\, \mathcal{E}_{2M}^{\rm poly}(f) \ge \sup_{M\ge N} \sqrt{\frac{M-N}{2M}}\, \mathcal{R}_M^{\rm equi}(f)\,; \tag{2}$$

$$f = f_{\rm harm} \Longrightarrow \mathcal{R}_{N^2}^{\rm equi}(f) << \mathcal{R}_N^{\rm free}(f) << \min_{M\ge N} \left( M e^{-\frac{N}{4\sqrt{M}}} \mathcal{R}_N^{\rm equi}(f) + \mathcal{R}_M^{\rm equi}(f) \right)\,. \tag{3}$$

**Corollary 1.** If $f = f_{\rm rad}$ i $\mathcal{E}_{2N}^{\rm poly}(f) \ne o\left(\mathcal{E}_N^{\rm poly}(f)\right)$, then $\mathcal{R}_N^{\rm free}(f) \ne o\left(\mathcal{R}_N^{\rm equi}(f)\right)$, $N \to \infty$. Thus, for radial functions the method $\mathcal{R}^{\rm free}$ is nor more efficient than $\mathcal{R}^{\rm equi}$.

This corollary strengthens some preceeding results due to V. E. Majorov [5] (further developments – in [6]) and V.N. Temlyakov [4] and the author [1]. It also confirms for $f = f_{\rm rad}$ a conjecture, formulated by D. Donoho and I. Johnstone [7].

**Corollary 2.** If $f = f_{\rm harm}$ and $\exists \varepsilon > 0$ such that $\mathcal{E}_{N^{2-\varepsilon}}^{\rm poly}(f) = o\left(\mathcal{E}_N^{\rm poly}(f)\right)$, then $\mathcal{R}_N^{\rm free}(f) = o\left(\mathcal{R}_N^{\rm equi}(f)\right)$. Thus, for harmonic functions the method $\mathcal{R}^{\rm free}$ is more effective in order, than $\mathcal{R}^{\rm equi}$. Moreover, if $f = f_{\rm harm}$ and $\exists \alpha > 0$ such that $\mathcal{R}_N^{\rm equi}(f) = O\left(N^{-\alpha}\right)$, then for $\forall \varepsilon > 0$ : $\mathcal{R}_N^{\rm free}(f) = o\left(N^{-2\alpha+\varepsilon}\right)$, i. e. $\mathcal{R}^{\rm free}$ is "almost square times" more efficient than $\mathcal{R}^{\rm equi}$.

This proves, that the part of the conjecture from [7] concerning harmonic functions fails to be true.

Ridge approximation of an individual function is dual with an infinite series of problems of Kolmogorov – Nikol'skii type, cf. [8], concerning optimization of quadrature formulas on compact classes of trigonometric polynomials. A quadrature formula with $N$ nodes on the period $[0, 2\pi)$ is a sampling linear functional of the form $Q_N[T] = Q\left(\{\vartheta_j\}_1^N, \{w_j\}_1^N\right)[T] := \sum_{j=1}^N w_j T(e^{i\vartheta_j})$; the global error $Q_N(T)$ in the recovery of a functional $F_a[T] := \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\vartheta}) T(e^{i\vartheta})\, d\vartheta$ on a certain class $\mathcal{K}$ of continuous periodic functions $T(e^{i\vartheta})$ is defined as the quantity

$$\mathcal{Q}\left(a, \mathcal{K};\, \{\vartheta_j\}_1^N, \{w_j\}_1^N\right) := \sup_{T\in\mathcal{K}} \left| \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\vartheta}) T(e^{i\vartheta})\, d\vartheta - \sum_{j=1}^N w_j T(e^{i\vartheta_j}) \right|.$$

In our case, the classes $\mathcal{K}$ are unit balls $\mathcal{K}_n := \{T \in \mathcal{T}_n,\ \|T, \mathcal{L}_{2\pi}^2\| \le 1\}$ in the subspaces $\mathcal{T}_n :=$ Span $\{e^{im\vartheta}\}_{-n}^n$ of trigonometric polynomials of $n$th order. Functionals $F_a[T]$ are generated by Fourier

– Radon analysis with respect to Chebyshev polynomials of the second kind (cf. [2], [9], [10]; $\boldsymbol{\theta} := \langle \cos\vartheta, \sin\vartheta \rangle$):

$$a(e^{i\vartheta}) = a_n(f, e^{i\vartheta}) := \int_{\mathbb{B}^2} f(\mathbf{x}) u_n(\mathbf{x}\cdot\boldsymbol{\theta})\, d\mathbf{x}, \quad u_n(x) := \frac{\sin{(n+1)\arccos x}}{\sqrt{\pi(1-x^2)}}, \quad n = 0, 1, \dots . \tag{4}$$

Basic identities which made it possible to "slice" the problem $\mathcal{R}_N^{\rm free}(f)$ into a series of optimization problems of Kolmogorov – Nikol'skii type, are contained in the next (sm. [1])

**Lemma 1** *Let $f(\mathbf{x}) \in \mathcal{L}^2(\mathbb{B}^2)$ and let $a_n(f, e^{i\vartheta})$ be defined by (4). Then*

$$f(\mathbf{x}) \stackrel{\mathcal{L}^2(\mathbb{B}^2)}{=} \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_1^\infty n a_{n-1}(f, e^{i\vartheta}) u_{n-1}(\mathbf{x}\cdot\boldsymbol{\theta})\right) d\vartheta, \ \left(\mathcal{E}_N^{\rm poly}(f)\right)^2 = \frac{1}{2\pi}\sum_{N+1}^\infty n \left\| a_{n-1}(f), \mathcal{L}^2_{2\pi}\right\|^2,$$
$$\left(\mathcal{R}_N^{\rm free}(f)\right)^2 = 2\pi \inf_{\{\vartheta_j\}_1^N} \sum_{N+1}^\infty n \inf_{\{w_j\}_1^N} \left(\mathcal{Q}\left(a_{n-1}(f), \mathcal{K}_{n-1};\, \{\vartheta_j\}_1^N, \{w_j\}_1^N\right)\right)^2 . \tag{5}$$

To realize the above scheme, a further concrete analysis is needed, of the arising Kolmogorov – Nikol'skii problems, for a given concrete sequence of the chebyshevian orthogonal momenta $a_n(f)$. In general, the momenta are trigonometric polynomials, $a_n(f) \in \mathcal{T}_n$, $a_n(f, -e^{i\vartheta}) \equiv (-1)^n a_n(f, e^{i\vartheta})$, and $2\pi\|f(\mathbf{x}), \mathcal{L}^2(\mathbb{B}^2)\|^2 = \sum_0^\infty (n+1)\|a_n(f), \mathcal{L}^2_{2\pi}\|^2$ (Parceval's identity).

The momenta are especially simple for radial and harmonic functions. If $f = f_{\rm rad}$, then $a_n(f)$ are constants (all odd momenta $a_{2n+1}(f) = 0$ in this case). If $f = f_{\rm harm}$, then $a_n(f, e^{i\vartheta}) = \alpha_n e^{in\vartheta} + \beta_n e^{-in\vartheta}$, i. e. pure oscillations of the highest possible frequency in $\mathcal{T}_n$. Further, the momenta of a planar wave function $F_\varphi(\mathbf{x}) = F(\mathbf{x}\cdot\boldsymbol{\varphi})$ are of the form $a_n(F_\varphi, e^{i\vartheta}) = \alpha_n \frac{\sin{(n+1)(\vartheta-\varphi)}}{\sin{(\vartheta-\varphi)}}$, i. e., numerical multiples of Dirichlet kernels. The latter is crucial for the duality of the type (5).

Let us consider the corresponding extremal problems of Kolmogorov – Nikol'skii on recovery of integrals of a trigonometric polynomial and its senior Fourier coefficient on the class $\mathcal{K}_n$:

$$\begin{aligned}
\mathcal{Q}_N^{\rm opt}(1, \mathcal{T}_n) &:= \inf_{Q_N} \sup_{T\in\mathcal{K}_n} \left|\frac{1}{2\pi}\int_0^{2\pi} T\left(e^{i\vartheta}\right) d\vartheta - Q_N[T]\right|, \\
\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta}, \mathcal{T}_n\right) &:= \inf_{Q_N} \sup_{T\in\mathcal{K}_n} \left|\frac{1}{2\pi}\int_0^{2\pi} T\left(e^{i\vartheta}\right) e^{-in\vartheta}\, d\vartheta - Q_N[T]\right|
\end{aligned} \tag{6}$$

The latter problem can be also reformulated as recovery of the values of analytic polynomials $P(z) = \sum_0^n c_m z^m \in \mathcal{P}_n^1$ at $z = 0$ via $N$ samples $P(z_j)$ taken on the circumference $|z| = 1$: one has $\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta},\mathcal{T}_n\right)=\mathcal{Q}_N^{\rm opt}\left(\mathcal{P}_{2n}^1\right)$, where

$$\mathcal{Q}_N^{\rm opt}\left(\mathcal{P}_n^1\right):=\inf_{\{z_j\}_1^N\subset\mathcal{S}^1,\,\{w_j\}_1^N}\ \sup_{P\in\mathcal{P}_n^1,\,P(e^{i\vartheta})\in\mathcal{K}_n}|P(0)-\sum_1^N w_jP(z_j)|. \tag{7}$$

The solutions of the problems concerning $\mathcal{Q}_N^{\rm opt}\left(1,\mathcal{T}_n\right)$ and $\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta},\mathcal{T}_n\right)$ turned out to be strikingly different.

The meaning of the next statement is in the following. First, it is not possible to recover, with a small error, the integrals of all polynomials from $\mathcal{K}_n$ , if the number of samples (available point data) $N$ is somewhat smaller than $n$, cf. also [16].

In contrast to it, recovery of the senior Fourier coefficient on $\mathcal{K}_n$ with a small global error is possible even in conditions of an "essential sampling deficiency". One needs only that $N$ be "much bigger" than $\sqrt{n}$.

We derive the upper estimate of $\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta},\mathcal{T}_n\right)$ by measuring not $N$ point values of polynomials, but rather a single value, say, at $\vartheta=0$, of an appropriately selected differential operator of order $N-1$. Namely, the setting of Kolmogorov – Nikol'skii problem admits exploiting of the closure of the set of quadrature formulas, and, respectively, that of the inequality $\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta},\mathcal{T}_n\right)\le\mathcal{Q}_N^{\rm col}\left(e^{-in\vartheta},\mathcal{T}_n\right)$, where

$$\mathcal{Q}_N^{\rm col}\left(e^{-in\vartheta},\mathcal{T}_n\right):=\min_{P\in\mathcal{P}_{N-1}^1}\ \sup_{T\in\mathcal{K}_n}\left|\frac{1}{2\pi}\int_0^{2\pi}T\left(e^{i\vartheta}\right)e^{-in\vartheta}\,d\vartheta-P\left(\frac{d}{d\vartheta}\right)T\left(e^{i\vartheta}\right)\Big|_{\vartheta=0}\right|. \tag{8}$$

Moreover, it is not hard to see, that this approach results in another problem of a classical chebyshevian nature – discrete method of the least squares for algebraic polynomial approximation:

$$\mathcal{Q}_N^{\rm col}\left(e^{-in\vartheta},\mathcal{T}_n\right)=\min_{P\in\mathcal{P}_{N-1}^1}\sqrt{\sum_{-n\le m<n}P^2(m)+(1-P(n))^2}. \tag{9}$$

**Theorem 2** *Let $n$, $N$ be natural numbers and $n>N$. Then*

$$\mathcal{Q}_N^{\rm opt}(1,\mathcal{T}_n)\sim\left(1-\frac{N}{n}\right)^{\frac12}. \tag{10}$$

*Further, there exist an absolute positive constant $c$ such that*

$$e^{-\frac{2N^2}{n}}<<\mathcal{Q}_N^{\rm opt}\left(e^{-in\vartheta},\mathcal{T}_n\right)\le\mathcal{Q}_N^{\rm col}\left(e^{-in\vartheta},\mathcal{T}_n\right)<<ne^{-\frac{N}{c\sqrt{n}}}. \tag{11}$$

In regard of (11) let us note that for $n > N$, not a single Fourier coefficient can be recovered by quadratures with $N$ equispaced nodes: the global error on $\mathcal{K}_n$ is $>> 1$. Apparently, the qualitative novelity of (11) compared with the known results of the theory of optimal quadratures (cf. [8], and also [11], [12]) is an improvement of the error estimates at the expence of the effect of "collapse" of the nodes. The latter means that in certain natural settings of the type (7), the optimal qadrature formula in Kolmogorov – Nikol'skii problem does not exist.

Preliminary estimates of discrete algebraic polynomial approximations on the right hand side of (9) appeared in discussions of the problem with my colleagues P. Petrushev, B. Popov and O. Trifonov at USC. Clearly, one can take in this problem an appropriately modified Chebyshev polynomial of the 1st kind and degree $N-1$. An upper estimate of the type (11) was communicated to the author by I. I. Sharapudinov. In a quite recent paper [13] it is proved that the righthand side of (9) is small if and only if $\frac{\sqrt{n}}{N} \to 0$.

The lower estimate of $\mathcal{Q}_N^{\mathrm{opt}}\left(e^{-in\vartheta}, \mathcal{T}_n\right)$ in (11) was recently established by the author in collaboration with B.S. Kashin during his visit to USC (April – May 98). It turned out that this estimate can be derived (cf. (7)) from the following stronger statement regarding polynomials with prescribed roots on the circumference $|z| = 1$.

**Lemma 2** *For each set $N$ of points $\{z_j\}_1^N$, $|z_j| = 1$ and a positive integer $m$, there is a polynomial $P(z) \in \mathcal{P}_{mN}^1$ satisfying $P(0) = 1$, $P(z_j) = 0$, $j = 1, 2, \ldots, N$, $\max_{|z|\le 1} |P(z)| \le e^{\frac{2N}{m}}$.*

This statement is a corollary of the solution, cf. [14], of the extremal problem of G. Hálász: find $\mu_m := \min_{P\in\mathcal{P}_m^{(1,0)}} \max_{|z|\le 1} |P(z)|$ where $\mathcal{P}_m^{(1,0)} := \{P(z) \in \mathcal{P}_m^1 : \;\; P(0) = 1, \;\; P(1) = 0\}$. For our purpose, the estimate $\mu_m \le 1 + \frac{2}{m}$ is sufficient (cf. [15], Ch. 5).

Lower estimates of $\mathcal{Q}_N^{\mathrm{opt}}(1, \mathcal{T}_n)$ and their multivariate analogs were studied by V.N. Temlyakov [16]. Basing on some results of B.S. Kashin [17] (cf. also [18]) it is established in [16] that if $N \le (1-\varepsilon)n$, where $\varepsilon > 0$, then $\mathcal{Q}_N^{\mathrm{opt}}(1, \mathcal{T}_n) \ge c_\varepsilon > 0$. This result was used in [1] for estimates of $\mathcal{R}^{\mathrm{free}}$ from below in the case of radial funcions.


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**Address.** K.I. Oskolkov, Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA;
e-mail: oskolkov@math.sc.edu; tel. (803)777-3776; fax: (803)777-3783.