Quantum Info Seminar

What is a quantum channel?

Represents some (any) quantum process that physically realizable (in principle).

Examples:

- A quantum computation
- Unitary evolution of an isolated system
- Measurements
- Q. & classical communication
- model errors or noise
- combinations of the above

Notation: A $\mathbb{C}$-space $\mathcal{H}$ is a finite dimensional Hilbert with an inner product $\langle \cdot , \cdot \rangle$ — positive definite $\langle u,u\rangle > 0$ if $u \neq 0$

$\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ — linear in 2nd arg — conjugate symmetric

and an orthonormal basis $\{e_1, \ldots, e_n\}$ $(n = \dim(\mathcal{H}))$

$$\langle e_i, e_j \rangle = \delta_{ij}$$

For $\mathbb{C}$-spaces $\mathcal{H}, \mathcal{J}$: $L(\mathcal{H}, \mathcal{J}) = \{A : \mathcal{H} \to \mathcal{J} \text{ is linear}\}$

$L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$

Adjoints: $A \in L(\mathcal{H}, \mathcal{J})$. $A^* \in L(\mathcal{J}, \mathcal{H})$ is unique map such that, $\forall u \in \mathcal{H}\ \forall v \in \mathcal{J}$, $\langle v, Au \rangle_{\mathcal{J}} = \langle A^* v, u \rangle_{\mathcal{H}}$

<u>Def</u>: A quantum channel (from $\mathcal{H}$ to $\mathcal{J}$) is a linear map $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ $[\Phi \in L(L(\mathcal{H}), L(\mathcal{J}))]$

that is

- trace-preserving
- completely positive

"trace preserving" means $tr(\Phi(A)) = tr\, A \quad \forall A \in L(\mathcal{H})$

<u>Def</u>: Φ is <u>positive</u> if $\forall A \in L(\mathcal{H})$, if $\underbrace{A \geq 0}_{\text{A is positive (semidef)}}$ then $\Phi(A) \geq 0$

Let $\mathcal{K}$ be any $\mathbb{C}$-space. Can form the linear map

$$\Phi \otimes \mathbb{1}_{L(\mathcal{K})} : L(\mathcal{H} \otimes \mathcal{K}) \to L(\mathcal{J} \otimes \mathcal{K})$$

defined so that $\forall A \in L(\mathcal{H})$ and $B \in L(\mathcal{K})$,

$$(\Phi \otimes \mathbb{1}_{L(\mathcal{K})})(A \otimes B) = \Phi(A) \otimes B$$

[If Φ is trace-preserving, then so is $\Phi \otimes \mathbb{1}_{L(\mathcal{K})}$ ← routine proof]

<u>Def</u>: $\Phi : L(\mathcal{H}) \to L(\mathcal{J})$ is <u>completely</u> <u>positive</u> if $\Phi \otimes \mathbb{1}_{L(\mathcal{K})}$ is positive for every $\mathbb{C}$-space $\mathcal{K}$.

<u>Example</u> of a positive operator that is not completely positive: transpose operator $[A \in \mathbb{C}^{2\times 2}]$

$$\Phi(A) = A^T$$

$\Phi \otimes \mathbb{1}_{\mathbb{C}^{2\times 2}}$ is <u>not</u> positive

$L(\mathcal{H}, \mathcal{J})$ is a $\mathbb{C}$-space! $\forall A, B \in L(\mathcal{H}, \mathcal{J})$

$$\langle A, B \rangle := tr(A^* B)$$

Examples:

Unitary Channel: $\Phi : L(\mathcal{H}) \to L(\mathcal{H})$

$$\Phi(A) = UAU^* \qquad [U \in L(\mathcal{H}) \text{ is unitary}]$$

describes any process on an isolated physical system. E.g. $\mathbb{1}_{L(\mathcal{H})}$ is a unitary channel ($U = \mathbb{1}_{\mathcal{H}}$) that causes no change (represents an ideal (noiseless) comm. channel, ~~&~~ or perfectly preserved memory).

Replacement Channel: Fix $\sigma \in L(\mathcal{J})$ density operator
$\sigma \geq 0$ & $tr\,\sigma = 1$

Define, for all $A \in L(\mathcal{H})$

$$\Phi(A) = (Tr\,A)\,\sigma \qquad [\text{so } \Phi(\rho) = \sigma \text{ any density operator } \rho]$$

State Preparation: $\Phi : \underbrace{L(\mathbb{C})}_{=\mathbb{C}} \to L(\mathcal{H})$ defined by

$\forall \alpha \in \mathbb{C}$, $\quad \Phi(\alpha) = \alpha \rho \quad$ [some fixed density operator ρ]

POVM (Positive operator-valued Measure):

Most general measurement on a quantum system where post-measurement state is ignored.

Set $\{M_1, \ldots, M_k\}$, where each $M_i \in L(\mathcal{H})$, $M_i \geq 0$ and $\sum_{i=1}^{k} M_i = \mathbb{1}_{\mathcal{H}}$.

The corresponding POVM is the channel $\Phi : L(\mathcal{H}) \to L(\mathbb{C}^k)$

$$\Phi(A) = \sum_{i=1}^{k} \mathrm{Tr}(AM_i) E_{ii} = \underbrace{\begin{bmatrix} \mathrm{Tr}(AM_1) & & 0 \\ & \ddots & \\ 0 & & \mathrm{Tr}(AM_k) \end{bmatrix}}_{\text{"classical" state}}$$

$$\left.\begin{aligned} \Phi &: L(\mathcal{H}_1) \longrightarrow L(\mathcal{J}_1) \\ \Psi &: L(\mathcal{H}_2) \longrightarrow L(\mathcal{J}_2) \end{aligned}\right\} \text{channels}$$

implies

$$\Phi \otimes \Psi : L(\mathcal{H}_1 \otimes \mathcal{H}_2) \to L(\mathcal{J}_1 \otimes \mathcal{J}_2)$$

is a channel

Trace Map: $\mathrm{Tr} : L(\mathcal{H}) \longrightarrow \overbrace{L(\mathbb{C})}^{=\mathbb{C}}$ is a channel

"destroy $\mathcal{H}$"
"discard $\mathcal{H}$"
"ignore $\mathcal{H}$"

Partial Trace: $\mathrm{Tr}_{\mathcal{K}} : \cancel{L(\mathcal{H})}\ L(\mathcal{H} \otimes \mathcal{K}) \longrightarrow L(\mathcal{H})$ $\quad$ $\mathcal{K}$ is a $\mathbb{C}$-space

$$\boxed{\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr}} : L(\mathcal{H} \otimes \mathcal{K}) \to L(\mathcal{H}) \quad \text{(channel)}$$

"ignore system $\mathcal{K}$"
"trace out $\mathcal{K}$"

$$(\mathbb{1}_{\mathcal{H}} \otimes \mathrm{Tr})(A \otimes B) = (\mathrm{tr}\, B) A \qquad \begin{aligned} A &\in L(\mathcal{H}) \\ B &\in L(\mathcal{K}) \end{aligned}$$